Features of the Nonlinear Fourier Transform for the dNLS Equation

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Für meine Großeltern

Zusammenfassung

Im ersten Teil dieser Arbeit erweitern wir die bekannte Spektraltheorie des Zakharov-Shabat Operators $L(\varphi) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_- \\ \varphi_+ & 0 \end{pmatrix}$, definiert auf dem Intervall $[0,1]$, für komplexwertige, 1-periodische Potentiale $\varphi = (\varphi_-, \varphi_+)$, die als Elemente des Fourier Lebesgue Raums FL^p, $1 \leqslant p < \infty$, gegeben sind, und beweisen asymptotische Abschätzung der periodischen und Dirichlet Eigenwerte in Abhängigkeit der Fourierkoeffizienten von φ. Mit Hilfe dieser Spektraltheorie dehnen wir anschliessend die Definition der Wirkungsvariablen $(I_n)_{n\in\mathbb{Z}}$ sowie der dazu kanonisch konjugierten Winkelvariablen $(\theta_n)_{n\in\mathbb{Z}}$ von L^2 auf FL^p, $p > 2$ aus. Diesen Variablen dienen im Anschluss als Grundlage für die Konstruktion reell analytischer Birkhoff Koordinaten auf FL^p.

Im zweiten Teil dieser Arbeit leiten wir, unter Verwendung der Birkhoff Koordinaten, eine neue Formel für die dNLS Frequenzen her, welche es erlaubt die Frequenzen analytisch auf FL^p, $p > 2$, fortzusetzen und ihr asymptotisches Verhalten präzise zu beschreiben. Ebenfalls leiten wir eine neue Formel für den dNLS Hamiltonian her, die im Anschluss dazu verwendet wird, den Hamiltonian nach geeigneter Renormalisierung reell analytisch auf FL^4_r zu erweitern. Wird der renormalisierte Hamiltonian in Wirkungsvariablen $I = (I_n)_{n\in\mathbb{Z}}$ ausgedrückt, so definiert er eine Funktion, die reell analytisch und strikt konkav in einer Umgebung von 0 im positiven Quadranten $\ell^2_+(\mathbb{Z})$ von $\ell^2(\mathbb{Z})$ ist. Abschliessend verwenden wir unsere Ergebnisse zu den Frequenzen, um das Anfangswertproblem von dNLS in Birkhoff Koordinaten zu studieren.

Im letzten Teil dieser Arbeit untersuchen wir die Birkhoff Abbildung in Sobolev Räumen hoher Regularität. Wir beweisen gleichmässige “tame” Abschätzungen aller ganzzahliger Sobolev Normen $\|\varphi\|_m$, $m \geqslant 1$, in Bezug auf gewichtete ℓ^2-Normen der Birkhoff Koordinaten und umgekehrt.

Abstract

In the first part of this thesis we extend the well-known spectral theory of the Zakharov-Shabat operator $L(\varphi) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_- \\ \varphi_+ & 0 \end{pmatrix}$, acting on the interval $[0,1]$, to the case where the potential $\varphi = (\varphi_-, \varphi_+)$ is a complex, 1-periodic element of the Fourier Lebesgue space FL^p, $1 \leqslant p < \infty$, and prove asymptotic estimates for its periodic and Dirichlet eigenvalues in terms of the Fourier coefficients of φ. The spectral theory is then used to extend the definition of the actions $(I_n)_{n\in\mathbb{Z}}$ and the canonically conjugated angles $(\theta_n)_{n\in\mathbb{Z}}$ from L^2 to FL^p, $p > 2$, which, in turn, are used to construct real analytic Birkhoff coordinates on FL^p.

In the second part of this thesis we derive, using the Birkhoff coordinates, a novel formula for the dNLS frequencies which allows to extend them analytically to FL^p, $p > 2$, and to characterize their asymptotic behavior. Similarly, we derive a formula for the dNLS Hamiltonian which is used to extend this Hamiltonian, after appropriate renormalization, real analytically to FL^4_r. When expressed in action variables $I = (I_n)_{n\in\mathbb{Z}}$, this renormalized Hamiltonian defines a function which is real analytic and strictly concave in a neighborhood of 0 in the positive quadrant $\ell^2_+(\mathbb{Z})$ of $\ell^2(\mathbb{Z})$. Finally, we use our previously obtained results on the frequencies to study the initial value problem of dNLS in Birkhoff coordinates.

In the final part of this thesis we investigate the Birkhoff map in Sobolev spaces of high regularity. We prove uniform tame estimates of all integer Sobolev norms $\|\varphi\|_m$, $m \geqslant 1$, in terms of weighted ℓ^2-norms of the Birkhoff coordinates and vice versa.

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Introduction

The *defocusing nonlinear Schrödinger equation* (dNLS)

$$\mathrm{i}\partial_t u = -\partial_x^2 u + 2|u|^2 u, \tag{0.1}$$

is a cubic perturbation of the linear Schrödinger equation for the wave function $u\colon \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ of a free particle evolving in one-space dimension. It turns out that this nonlinear dispersive evolution equation is a universal model describing slowly varying wave envelopes in dispersive media and arises in various physical systems such as water waves, plasma physics, solid-state physics, and nonlinear optics. A central feature of the dNLS equation in one space dimension is that the linear dispersion, which tends to spread the wave packet, is balanced by the cubic nonlinearity describing the self interaction of the wave with itself. It led to far reaching applications of the dNLS equation to optical telecommunications [61].

In this thesis we consider the dNLS equation (0.1) with periodic boundary conditions $u(t,1) = u(t,0)$. It is well known that in this set up, it can be written as a Hamiltonian system

$$\mathrm{i}\partial_t u = \partial_{\bar{u}}\mathcal{H},$$

with Hamiltonian

$$\mathcal{H}(u) = \int_0^1 (|\partial_x u|^2 + |u|^4)\,\mathrm{d}x.$$

Instigated by the work of Gardner et al. [19] and Lax [52] for the KdV equation, Zakharov & Shabat [66] discovered that the dNLS flow defines an isospectral deformation of the Zakharov-Shabat operator, with u playing the role of a potential, and showed in this way that dNLS admits infinitely many integrals in involution. Starting from those integrals, Kappeler and collaborators have constructed, in a series of works culminating in [23], a globally defined nonlinear normal form transformation $u \mapsto (x_n, y_n)_{n\in\mathbb{Z}}$, known as *Birkhoff map*. When expressed in these new coordinates, the NLS Hamiltonian is a real analytic function of the actions $I_n = (x_n^2 + y_n^2)/2$, $n\in\mathbb{Z}$, alone and, in turn, the dNLS flow is transformed into an infinite chain of nonlinearly coupled oscillators

$$\partial_t x_n = -\omega_n y_n, \quad \partial_t y_n = \omega_n x_n, \qquad n\in\mathbb{Z}.$$

The action variables are preserved along the flow lines and so are the *frequencies* ω_n, $n\in\mathbb{Z}$, which are given by $\omega_n = \partial_{I_n}\mathcal{H}$. In these coordinates, it becomes evident that every x-periodic solution of dNLS is periodic, quasi-periodic, or almost-periodic in time. The principal purpose for constructing such coordinates is the study of perturbations of the dNLS equation. In view of the nonlinearity of the Birkhoff map a drawback is that when perturbed equations are expressed in these coordinates, many of their features, which persist under linear transformations such as the Fourier transform, are not readily available in these coordinates. To be useful for applications it is therefore important to describe the Birkhoff map as well as the Hamiltonian, when expressed in these coordinates, as detailed as possible. The purpose of this thesis is to contribute to the analysis of the Birkhoff map for completely integrable PDEs. In order to make this thesis not too long, we restrict here to the case of the dNLS equation. We have obtained corresponding results for the KdV equation, and we believe that our methods apply to other integrable PDEs as well.

The Birkhoff map appears as a nonlinear perturbation of the Fourier transform, and one may view the coordinates x_n, y_n, $n \in \mathbb{Z}$, as nonlinear Fourier coefficients depending real analytically on u. In fact, the derivative of the Birkhoff map at the origin is the Fourier transform, and, as for the Fourier transform, it admits Parseval's identity [55, 23] asserting that the ℓ^2-norm of the nonlinear Fourier coefficients equals the L^2-norm of u. The first main result of this thesis provides uniform tame estimates of all integer Sobolev norms $\|u\|_{H^m}$, $m \geqslant 1$, in terms of weighted ℓ^2 norms of the nonlinear Fourier coefficients and vice versa.

The second main result of this thesis concerns convexity properties of the dNLS Hamiltonian. For finite dimensional Hamiltonian systems with convex Hamiltonians, the Nekhoroshev theory allows to infer long time stability of the system under perturbations. It is expected that convexity properties of the Hamiltonian will also play an important role for developing a corresponding theory for perturbations of infinite dimensional integrable systems. The dNLS Hamiltonian, viewed as a function of the actions, is real analytic on a certain weighted ℓ^1 space. We prove that, after an appropriate renormalization, this Hamiltonian extends to a function which is real analytic and strictly concave in a neighborhood of the origin in ℓ^2.

The third main result of this thesis concerns the diffeomorphism property of the actions to frequencies map – or frequency map for short – for dNLS. In perturbation theory, having precise control over the frequencies is crucial to handle resonances. We prove that after an appropriate renormalization, the frequency map $(I_n)_{n\in\mathbb{Z}} \mapsto (\omega_n^\star(I))_{n\in\mathbb{Z}}$ extends for any $2 < p < \infty$ to a real analytic local diffeomorphism of a neighborhood of the origin in $\ell^{p/2}$. This allows one to use in perturbation theory the actions as internal parameters to adjust the frequencies.

To obtain the third main result, we have to extend the Birkhoff map to phase spaces with corresponding actions in $\ell^{p/2}$ for $2 < p < \infty$. Since, at first order, the actions are proportional to the squared modulus of the Fourier coefficients, these phase spaces are the *Fourier Lebesgue spaces* FL^p, i.e. the spaces of periodic distributions whose Fourier coefficients are in ℓ^p. To prove the second result we then use that the Birkhoff map extends analytically to FL^4. Note that $FL^2 = L^2$, while for any $p > 2$, the space FL^p contains elements, which are periodic distributions, but not more singular than those in $H^{-1/2}$ since $FL^p \hookrightarrow H^{-1/2}$ for any $2 < p < \infty$. Here, $s = -1/2$ is the critical regularity for the dilation symmetry of the dNLS equation on $H^s(\mathbb{R})$, $u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x)$. Further, the Fourier Lebesgue spaces $FL^p(\mathbb{R})$ are invariant under the Galilean symmetry of the dNLS equation $u(t,x) \mapsto \mathrm{e}^{\mathrm{i}\nu x - \mathrm{i}\nu^2 t} u(t, x - 2\nu t)$.

Statement of main results

To state our main results, we write the dNLS equation (0.1) as an *infinite dimensional Hamiltonian system* on the phase space $H^s_c \coloneqq H^s(\mathbb{T},\mathbb{C}) \times H^s(\mathbb{T},\mathbb{C})$, where $s \geqslant 0$ and $\mathbb{T} \coloneqq \mathbb{R}/\mathbb{Z}$. The elements of this phase space are denoted by $\varphi = (\varphi_-, \varphi_+)$ and we endow this space with the Poisson bracket

$$\{F, G\} \coloneqq -\mathrm{i} \int_{\mathbb{T}} (\partial_{\varphi_-} F \, \partial_{\varphi_+} G - \partial_{\varphi_+} F \, \partial_{\varphi_-} G) \, \mathrm{d}x.$$

Here $\partial_{\varphi_\pm} F$ and $\partial_{\varphi_\pm} G$ denote the components of the L^2-gradients $\partial_\varphi F$ and $\partial_\varphi G$ of C^1-functionals F and G with respect to the bilinear inner product

$$\langle \varphi, \psi \rangle_r \coloneqq \int_{\mathbb{T}} (\varphi_+ \psi_+ + \varphi_- \psi_-) \, \mathrm{d}x.$$

On H^s_c, $s \geqslant 1$, the *NLS Hamiltonian system* with Hamiltonian

$$\mathcal{H}(\varphi_-, \varphi_+) := \int_{\mathbb{T}} (\partial_x \varphi_- \partial_x \varphi_+ + \varphi_-^2 \varphi_+^2) \, \mathrm{d}x$$

is then given by

$$\mathrm{i}\partial_t (\varphi_-, \varphi_+) = (\partial_{\varphi_+} \mathcal{H}, -\partial_{\varphi_-} \mathcal{H}).$$

The dNLS equation is obtained by restricting the NLS system to the invariant subspace of *real type* states

$$H^s_r := \{\varphi \in H^s_c : \varphi^* = \varphi\}, \qquad \varphi^* = (\overline{\varphi_+}, \overline{\varphi_-}).$$

Indeed, with $\varphi = (u, \overline{u})$ we get

$$\mathrm{i}\partial_t u = \mathrm{i}\{u, \mathcal{H}\} = \partial_{\overline{u}} \mathcal{H}(u, \overline{u}) = -\partial_x^2 u + 2|u|^2 u.$$

Written in this form, the dNLS equation is known to be *completely integrable* by the inverse scattering method [66]. Actually, according to [23], it is integrable in the strongest possible sense meaning that it admits global *Birkhoff coordinates* $(x_n, y_n)_{n\in\mathbb{Z}}$. To give a precise statement, let us introduce the model space $h^s_r := \ell^{s,2}_{\mathbb{R}} \times \ell^{s,2}_{\mathbb{R}}$ where $\ell^{s,p}_{\mathbb{R}} := \ell^{s,p}(\mathbb{R}, \mathbb{Z})$. This space is endowed with the canonical Poisson bracket $\{x_n, y_m\} = -\delta_{n,m}$ while all other brackets vanish. The *Birkhoff map* $\varphi \mapsto (x_n, y_n)_{n\in\mathbb{Z}}$ is a bi-analytic, canonical diffeomorphism $\Omega\colon H^0_r \to h^0_r$, whose restriction $\Omega|_{H^m_r}\colon H^m_r \to h^m_r$ is also bi-analytic for any integer $m \geqslant 1$, and which has the property that the transformed NLS Hamiltonian $\mathcal{H} \circ \Omega^{-1}$ is a real analytic function of the actions $I_n = (x_n^2 + y_n^2)/2$, $n \in \mathbb{Z}$, alone. More precisely, since the NLS Hamiltonian $\mathcal{H}$ is defined on H^1_r, the function $H \equiv \mathcal{H} \circ \Omega^{-1}$ is real analytic on the positive cone $\ell^{2,1}_+$ of $\ell^{2,1}_{\mathbb{R}}$, where for any $s \geqslant 0$ and $1 \leqslant q \leqslant \infty$,

$$\ell^{s,q}_+ := \{I \in \ell^{s,q} : I_m \geqslant 0 \quad \text{for all} \quad m \in \mathbb{Z}\}.$$

As a consequence, the equations of motion in Birkhoff coordinates in h^1_r are given by

$$\partial_t x_n = -\omega_n y_n, \quad \partial_t y_n = \omega_n x_n, \qquad \omega_n := \partial_{I_n} H. \tag{0.2}$$

One may thus think of the Birkhoff map Ω as a nonlinear Fourier transform for the dNLS equation. Indeed, the derivative $\mathrm{d}_0\Omega$ of Ω at the origin is a version of the Fourier transform and on $L^2_r \equiv H^0_r$, as for the Fourier transform, we have Parseval's identity

$$\|\Omega(\varphi)\|_{\ell^2_r} = \|\varphi\|_{L^2_r},$$

– see [55, 23]. Our first main result says that for higher order Sobolev norms, the following version of Parseval's identity holds for the nonlinear map Ω.

Theorem 0.1 *For any integer $m \geqslant 1$, there exist absolute constants c_m, $d_m > 0$ such that the restriction of Ω to H^m_r satisfies the two sided estimates*

(i) $\|\Omega(\varphi)\|_{h^m_r} \leqslant c_m \big(\|\varphi\|_{H^m_r} + (1 + \|\varphi\|_{H^1_r})^{2m} \|\varphi\|_{H^0_r}\big)$,

and

(ii) $\|\varphi\|_{H^m_r} \leqslant d_m \big(\|\Omega(\varphi)\|_{h^m_r} + (1 + \|\Omega(\varphi)\|_{h^1_r})^{4m-3} \|\Omega(\varphi)\|_{h^0_r}\big)$. ⋊

The main features of Theorem 0.1 are that the constants c_m and d_m can be chosen independently of φ and that the estimate (i) is linear in the highest Sobolev norm $\|\varphi\|_{H_r^m}$ for $m \geqslant 2$, whereas the estimate (ii) is linear in the highest h_r^m-norm $\|\Omega(\varphi)\|_{h_r^m}$ of $\Omega(\varphi)$. Note that, the remainders in estimates (i) and (ii) involve only H_r^1 and h_r^1-norms respectively.

Our second main result is concerned with the convexity properties of the NLS Hamiltonian $H\colon \ell_+^{2,1} \to \mathbb{R}$. According to [51], H admits an expansion at $I = 0$ of the form

$$H = \sum_{m\in\mathbb{Z}} (2m\pi)^2 I_m + 2H_1^2 - \sum_{m\in\mathbb{Z}} I_m^2 + \cdots, \qquad H_1 = \sum_{m\in\mathbb{Z}} I_m,$$

where the dots stand for higher order terms in I. In particular, at $I = 0$,

$$\partial_{I_n} H\big|_{I=0} = (2n\pi)^2, \qquad \partial_{I_n}\partial_{I_m} H\big|_{I=0} = -2\delta_{nm}, \qquad n, m \in \mathbb{Z}.$$

Consequently, the frequencies $\omega_n = \partial_{I_n} H$ have an expansion at $I = 0$ of the form – see also [22, Corollary 3.2]

$$\omega_n = (2n\pi)^2 + 4H_1 - 2I_n + \cdots. \tag{0.3}$$

The Hamiltonian $H_1 = \sum_{m\in\mathbb{Z}} I_m$, which appears as a first order term in the expansion of the frequencies, clearly cannot be extended to $\ell_+^{p/2}$ with $p > 2$. To this end, we introduce the frequencies $\omega_n^\star$ where we remove from ω_n the part which is constant on level sets of H_1,

$$\omega_n^\star = \omega_n - (2n\pi)^2 - 4H_1.$$

These frequencies have an expansion at $I = 0$ of the form $\omega_n^\star = -2I_n + \cdots$, and correspond to the Hamiltonian

$$H^\star = H - 2H_1^2 - \sum_{m\in\mathbb{Z}} (2m\pi)^2 I_m,$$

which has an expansion at $I = 0$ of the form $H^\star = -\sum_{m\in\mathbb{Z}} I_m^2 + \cdots$.

Motivated by the perturbation theory of the dNLS equation, Korotyaev [47] conjectured – see also Korotyaev & Kuksin [49] – that the Hamiltonian $H^\star$ admits a real analytic extension to ℓ_+^2 and is strictly concave there. We prove this conjecture in a neighborhood of the origin.

Theorem 0.2 *The Hamiltonian $H^\star$ is real analytic and nonpositive on ℓ_+^1, and vanishes only at $I = 0$. Moreover, $H^\star$ extends real analytically to a neighborhood of 0 in ℓ_+^2 and is strictly concave there in the sense that for all I of that neighborhood,*

$$\mathrm{d}_I^2 H^\star(J, J) \leqslant -\sum_{m\in\mathbb{Z}} J_m^2, \qquad \forall\ J = (J_m)_{m\in\mathbb{Z}} \in \ell_{\mathbb{R}}^2. \quad \rtimes$$

Remark. Actually, we prove that $H^\star$ extends real analytically to an open subset $\mathcal{V}^2$ of $\ell_{\mathbb{C}}^2$ which contains ℓ_+^1. In particular, $\mathcal{V}^2 \cap \ell_+^2$ is an open and dense subset of ℓ_+^2. It is an open question whether $\mathcal{V}^2$ contains all of ℓ_+^2 and whether $H^\star$ is strictly convex on all of ℓ_+^2. $\multimap$

The *frequency map* $\omega^\star = \partial H^\star = (\omega_n^\star)_{n\in\mathbb{Z}}$, a priori a real analytic function $\omega^\star\colon \ell_+^{2,1} \to \ell_{\mathbb{R}}^{-2,\infty}$, admits in view of Theorem 0.2 a real analytic extension $\omega^\star\colon \mathcal{V}^2 \to \ell_{\mathbb{C}}^2$ and is a local diffeomorphism around $I = 0$. As a consequence, the Hamiltonian $H^\star$ does not admit a C^1-extension to any neighborhood U^q of the origin in ℓ_+^q for any $q > 2$ since this would imply that $\omega^\star(I) = \partial H^\star(I) \in \ell_{\mathbb{R}}^{q'}$ for all I in U^q where $1/q + 1/q' = 1$. This, however, is impossible due to the diffeomorphism property. On the other hand, it is left open whether the frequency map can be extended further. Our third main result says that this is indeed the case.

Theorem 0.3 *The map $\omega^\star$ is defined on ℓ^1_+, takes values in $\bigcap_{r>1}\ell^r_{\mathbb{R}}$, and $\omega^\star\colon \ell^1_+ \to \ell^r_{\mathbb{R}}$ is real analytic for any $r > 1$. Moreover, for any $p > 2$, there exists an open neighborhood $\mathcal{V}^{p/2}$ of 0 in $\ell^{p/2}_{\mathbb{C}}$ so that $\omega^\star$ admits a real analytic extension $\omega^\star\colon \mathcal{V}^{p/2} \to \ell^{p/2}_{\mathbb{C}}$ which is a diffeomorphism onto its image and satisfies the asymptotic estimates*

$$\omega^\star_n + 2I_n = \begin{cases} \ell^{1+}_n, & 2 < p \leqslant 3 \\ \ell^{p/3}_n, & p > 3. \end{cases} \quad \rtimes$$

Remark. We actually prove that the domain of analyticity $\mathcal{V}^{p/2}$ of $\omega^\star$ can be chosen to contain ℓ^1_+, that $\omega^\star$ is a local diffeomorphism generically on $\mathcal{V}^{p/2}$, and that the asymptotic estimates hold locally uniformly on $\mathcal{V}^{p/2}$. Note that $\mathcal{V}^{p/2} \cap \ell^{p/2}_+$ is an open and dense subset of $\ell^{p/2}_+$. It is an open question whether $\mathcal{V}^{p/2}$ contains all of $\ell^{p/2}_+$ for any $p > 2$. $\multimap$

The proof of Theorem 0.3 is based on an extension of the Birkhoff map to the Fourier Lebesgue spaces $FL^p_r \coloneqq \{\varphi \in FL^p_{\mathbb{C}} \times FL^p_{\mathbb{C}} : \varphi^* = \varphi\}$ with $2 < p < \infty$, and the one of Theorem 0.2 on an extension to FL^4_r. Here, $FL^p_{\mathbb{C}} = FL^p_{\mathbb{C}}(\mathbb{T},\mathbb{C})$ denotes the space of 1-periodic complex-valued distributions whose Fourier coefficients are in ℓ^p.

The fact that the Hamiltonian H_1 appears as a first order term in the expansion of the dNLS frequencies (0.3) indicates that the dNLS solution map cannot be continuously extended to FL^p_r for $p > 2$. Instead, one considers the *renormalized NLS frequencies*

$$\omega^r_n \coloneqq \omega_n - 4H_1 = (2n\pi)^2 + \omega^\star_n$$

corresponding to the Hamiltonian system

$$\mathrm{i}\partial_t(\varphi_-,\varphi_+) = (\partial_{\varphi_+}\mathcal{H}^r, -\partial_{\varphi_-}\mathcal{H}^r), \qquad \mathcal{H}^r = \mathcal{H} - 2\mathcal{H}_1^2.$$

When restricted to real type states $\varphi = (v,\overline{v})$, this Hamiltonian system admits the form of the *renormalized dNLS equation* $(\text{dNLS})_r$ also called *Wick-ordered NLS*

$$\mathrm{i}\partial_t v = \partial_{\overline{v}}\mathcal{H}^r = -\partial_x^2 v + 2|v|^2 v - 4\left(\int_{\mathbb{T}} |v|^2\,\mathrm{d}x\right)v. \tag{0.4}$$

Since the dNLS flow preserves the L^2-norm of any initial datum on H^s_r, $s \geqslant 0$, we have for any dNLS solution $u(t)$ with initial datum u_0 that $v(t) = \mathrm{e}^{\mathrm{i}\|u_0\|_2 t}u(t)$ is a solution of $(\text{dNLS})_r$ with the same initial datum. On H^s_r, $s \geqslant 0$, we thus can freely convert solutions of (0.1) into ones of (0.4) and vice versa. Theorem 0.3 applies to the study of the solution map of the $(\text{dNLS})_r$ equation in FL^p_r, $2 < p < \infty$, constructed by Christ [11] and Grünrock & Herr [25] – see Section 22.

ℓ^p-Notation. The following notation is used throughout this work. A sequence of complex numbers $(a_n)_{n\in\mathbb{A}}$ is denoted $a_n = \ell^p_n + \ell^q_n$ if it can be decomposed as $a_n = x_n + y_n$ with $(x_n) \in \ell^p(\mathbb{A})$ and $(y_n) \in \ell^q(\mathbb{A})$. Moreover, $a_n = \ell^{1+}_n$ means that $(a_n) \in \ell^r(\mathbb{A})$ for any $r > 1$.

Method of proof. Theorem 0.1 is obtained from corresponding estimates of the action variables

$$\|I(\varphi)\|_{\ell^{2m,1}} \leqslant c_m^2(\|\varphi\|^2_{H^m_r} + (1 + \|\varphi\|_{H^1_r})^{4m}\|\varphi\|^2_{H^0_r}),$$

and

$$\|\varphi\|^2_{H^m_r} \leqslant d_m^2(\|I(\varphi)\|_{\ell^{2m,1}} + (1 + \|I(\varphi)\|_{\ell^{2,1}})^{4m-3}\|I(\varphi)\|_{\ell^1}).$$

To prove the latter, we consider the action variables $J_{n,k}$ on levels $k \geqslant 1$, introduced for different purposes by McKean & Vaninsky [55]. They can be defined entirely in terms of the periodic spectrum of the associated *Zakharov-Shabat operator* $L(\varphi)$ which appears in the Lax-pair formulation of the NLS equation,

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & \varphi_- \\ \varphi_+ & 0 \end{pmatrix}.$$

If $\varphi \in H_r^0$, then periodic spectrum of $L(\varphi)$ is well known to be real and discrete. The periodic eigenvalues are preserved by the dNLS flow, and can be ordered, when listed with multiplicities, as a bi-infinite sequence $(\lambda_n^\pm)_{n\in\mathbb{Z}}$ so that

$$\cdots \leqslant \lambda_{n-1}^+ < \lambda_n^- \leqslant \lambda_n^+ < \lambda_{n+1}^- \leqslant \cdots, \qquad \lambda_n^\pm = n\pi + \ell_n^2.$$

The asymptotic behavior of the actions on odd levels $k = 2m+1$ turns out to be

$$J_{n,2m+1} \sim (\lambda_n^\pm)^{2m} I_n \sim (n\pi)^{2m} I_n, \qquad |n| \to \infty.$$

Moreover, they satisfy the trace formula

$$\sum_{n\geqslant 1} J_{n,2m+1} = \frac{1}{4^m}\mathcal{H}_{2m+1}, \qquad m \geqslant 1,$$

where $\mathcal{H}_{2m+1}$ denotes the $(2m+1)$th Hamiltonian in the NLS hierarchy,

$$\mathcal{H}_1 = \int_{\mathbb{T}} |\psi|^2 \,\mathrm{d}x, \quad \mathcal{H} = \mathcal{H}_3 = \int_{\mathbb{T}} (|\psi'|^2 + |\psi|^4)\,\mathrm{d}x, \quad \ldots.$$

In general, on H_r^m, for $m \geqslant 1$,

$$\mathcal{H}_{2m+1} = \int_{\mathbb{T}} \left(|\psi^{(m)}|^2 + p_m(\psi,\ldots,\psi^{(m-1)}) \right) \mathrm{d}x,$$

where p_m is a polynomial expression in ψ and its first $m-1$ derivatives. Viewing $\mathcal{H}_{2m+1}$ as a lower order perturbation of the H_r^m-norm, we obtain at first order

$$\sum_{n\in\mathbb{Z}} (n\pi)^{2m} I_n \sim \sum_{n\in\mathbb{Z}} J_{n,2m+1} \sim \|\varphi\|_m^2.$$

The essential ingredient to make this approach work is a sufficiently accurate localization of the periodic eigenvalues $\lambda_n^\pm$ of φ in H_c^1 for all $n \in \mathbb{Z}$ with $|n|$ above a certain threshold depending only on $\|\varphi\|_{H^1}$. Above this threshold we can directly compare the weighted action norms and the polynomial expressions in φ as described above, while the remainder for $|n|$ below the threshold can be regarded as an H^1-error term. In this way, we obtain the claimed estimate of the action variables which imply Theorem 0.1. Note that our method of proof completely avoids the use of auxiliary spectral quantities such as *spectral heights* or *conformal mapping theory*.

The discriminant $\Delta(\lambda,\varphi)$, which for φ sufficiently regular is defined as the trace of the fundamental solution of the operator $L(\varphi)$, admits an asymptotic expansion at $\lambda = +\infty$ involving all the Hamiltonians of the NLS hierarchy. Inspired from the work in [48, 27], we derive a novel formula for the Hamiltonian $H^\star$ from this expansion using the residue calculus. This renders $H^\star$ as the infinite sum $\frac{4}{3}\sum_{k\in\mathbb{Z}} \mathcal{R}_k^{(3)}$, where the functionals $\mathcal{R}_k^{(3)}$ are defined only in terms of $\Delta(\lambda,\varphi)$. Since the latter is real analytic on FL_r^p for any $1 < p < \infty$, so are the functionals $\mathcal{R}_k^{(3)}$. Moreover, their asymptotic behavior turns out to be $\mathcal{R}_k^{(3)} = O(|\lambda_k^+ - \lambda_k^-|^4)$, hence their sum is absolutely and locally uniformly convergent to a real analytic function on FL^4_r. Since the actions of FL^4_r are an open subset of ℓ^2_+ which contains ℓ^1_+, Theorem 0.2 follows.

In a similar fashion, we derive a novel formula for the frequencies $\omega_n^\star$ from the expansion of $\Delta(\lambda,\varphi)$ at infinity and the residue calculus, rendering $\omega_n^\star$ as the infinite sum $-\frac{4}{2\pi}\sum_{k\in\mathbb{Z}}\Omega_{nk}^{(2)}$. Again, the functionals $\Omega_{nk}^{(2)}$ are defined only in terms of $\Delta(\lambda,\varphi)$ and hence are real analytic on FL^p_r for any $1<p<\infty$. This time one deduces from the asymptotic behavior of the periodic eigenvalues that the sum is convergent to a real analytic function on FL^p_r for any $1<p<\infty$ and satisfies the asymptotic estimates as claimed in Theorem 0.3.

Related results. Theorem 0.1 for the case $m=1$ was proved by Korotyaev [45] using conformal mapping theory, see also [48]. However, his method does not seem applicable for the case $m\geqslant 2$. In fact, it is stated as an open problem in [48]. The case $m\geqslant 2$ was first considered by the author in [59]. Theorem 0.1 improves on the estimates obtained there by showing that the remainders of the bounds in (i) and (ii) involve only H^1- respectively h^1-norms instead of H^{m-1}- respectively h^{m-1}-norms. For the case of the KdV equation, Korotyaev [44, 46] obtained polynomial bounds of the Sobolev norms $\|u\|_{H^m}$ in terms of the action variables where the order of the polynomials grows factorially in m. Note that the bound in (ii) of Theorem 0.1 is of order one in the Sobolev norm $\|\varphi\|_{H^m_r}$ and that the exponent of the remainder grows linearly in m and only involves the H^1 norm of φ. It turns out that our method can also be applied to the KdV equation. In [60] we improve on the bounds obtained by Korotyaev in [44, 46].

One can view the action I_n as an (actually 1-smoothing) perturbation of the squared modulus of the nth Fourier coefficient. Our method of comparing the weighted action norms with the Hamiltonians of the NLS hierarchy amounts to a separate analysis of Fourier modes of low and high frequencies. This idea has a long history in the analysis of nonlinear PDEs. Most recently, it led Colliander, Keel, Staffilani, Takaoka & Tao [65, 14, 15, 16] to invent the I-Method, which allows to obtain global wellposedness of subcritical equations in low regularity regimes where the Hamiltonian (or other integrals) of the equation cease to be well defined. Their idea is to damp all sufficiently high Fourier modes of a local solution such that the Hamiltonian can be controlled by weaker norms while still being an »almost conserved« quantity. In order to achieve this, one has to choose the damping subtle enough such that the nonlinearity of the equation does not create a significant interaction of low and high frequencies. Our situation for the proof of Theorem 0.1 is so to say reversed to that of the I-Method: Since we aim for quantitative global estimates, controlling the modes of low frequencies is the most delicate part. Here the localization of the periodic eigenvalues of the Zakharov-Shabat operator associated with the NLS equation plays a crucial role.

The Hamiltonian $H^\star$ has been shown by Korotyaev [47] to extend continuously to a nonpositive function on ℓ^1_+ satisfying two sided estimates in the ℓ^2-norm, leaving the question of analytic extension and strict convexity open. For the case of the KdV Hamiltonian we refer to [49] where a similar result is obtained. Moreover, in [27] the strict convexity of the renormalized KdV Hamiltonian has been proved in a neighborhood of the origin of ℓ^2_+.

The asymptotic behavior of the dNLS frequencies has been considered by Grébert & Kappeler [22] on H^1_r where it is shown that $\omega_n=(2n\pi)^2+O(1)$, and by Korotyaev [47] who shows that the frequencies $\omega_n^\star$ extend to continuous functions on L^2_r with asymptotic behavior $\omega_n^\star=\ell^2_n$ leaving the question of analytic extension open. Note that we prove the more detailed asymptotics

$\omega_n^\star = \ell_n^{p/2} + \ell_n^{1+}$ on FL_r^p and establish the analyticity as well as the diffeomorphism property of the frequency map $\omega^\star$ near $I = 0$. The frequency map of KdV has been studied intensely for finite gap potentials – see [5, 6, 50, 33]. In [27], the renormalized KdV frequency map has been proved to be a local diffeomorphism at the origin of ℓ_+^2 by extending the renormalized KdV Hamiltonian to this space and taking into account the expansion of the frequencies near $I = 0$. Subsequently, we show in [32] in analogy to Theorem 0.3 that the diffeomorphism property of the renormalized KdV frequencies holds generically and on a larger scale of spaces by extending the frequency map directly.

Organization of this work. In Chapter 1 we extend the definition of the Zakharov-Shabat operator $L(\varphi)$ and its spectral theory to the Fourier Lebesgue spaces FL_c^p, $p > 2$, and prove asymptotic estimates of the periodic and Dirichlet eigenvalues in terms of the Fourier coefficients of φ.

The spectral theory is then used in Chapter 2 to extend the definition of action and canonically conjugated angle variables from L^2 to FL^p with $p > 2$. In Chapter 3, these action angle variables are used to construct real analytic Birkhoff coordinates on FL_r^p.

In Chapter 4 we derive a novel formula for the NLS frequencies which allows to extend them analytically and to prove the asymptotic behavior claimed in Theorem 0.3. Similarly, we derive a formula for the NLS Hamiltonian which is used to extend $H^\star$ real analytically to FL_r^4. The convexity properties claimed in Theorem 0.2 then follow from the diffeomorphism properties of the frequencies. Finally, we use our previously obtained results on the frequencies to study the initial value problem of (dNLS)$_r$ in Birkhoff coordinates.

In the final Chapter 5, we investigate the Birkhoff map in Sobolev spaces of high regularity. We first establish a localization of the Zakharov-Shabat spectrum which is uniform on bounded subsets of H_c^1. Subsequently, this localization together with some technical results obtained in Chapter 4 is used to obtain a quantitative version of the first order estimate $\sum_{n\in\mathbb{Z}}(n\pi)^{2m} I_n \sim \sum_{n\in\mathbb{Z}} J_{n,2m+1} \sim \|\varphi\|_m^2$. This proves Theorem 0.1.

For the convenience of the reader we added several appendices providing auxiliary results which might be of independent interest. There is also a list of frequently used notations at the very end.

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Chapter 1

Spectral theory

1. Overview

Let $L^2_{\mathbb{C}} \coloneqq L^2(\mathbb{T},\mathbb{C})$ denote the space of 1-periodic complex-valued L^2-functions and define $L^2_c \coloneqq L^2_{\mathbb{C}} \times L^2_{\mathbb{C}}$. Consider for a *potential* $\varphi = (\varphi_-, \varphi_+)$ taken from L^2_c the *Zakharov-Shabat operator*

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}, \tag{1.1}$$

acting as an unbounded operator on L^2_c with domain

$$\mathcal{D}_{per} = \{ f \in L^2_c : f \in H^1_{\text{loc}} \text{ and } f(1) = \pm f(0) \}.$$

The spectrum of $L(\varphi)$ is well known to consist of a double infinite sequence of eigenvalues $(\lambda_n^\pm)_{n\in\mathbb{Z}}$ which can be ordered lexicographically – first by their real and second by their imaginary part – so that when listed with multiplicities

$$\cdots \preccurlyeq \lambda_{n-1}^+ \preccurlyeq \lambda_n^- \preccurlyeq \lambda_n^+ \preccurlyeq \lambda_{n+1}^- \preccurlyeq \cdots, \qquad \lambda_n^\pm = n\pi + \ell_n^2.$$

Here, ℓ_n^2 denotes a generic ℓ^2-sequence. By a slight abuse of notation we call the spectrum of $L(\varphi)$ the *periodic spectrum of* φ.

When the potential φ is of *real type*, that is $\varphi_- = \overline{\varphi_+}$, then $L(\varphi)$ is self-adjoint hence the periodic spectrum is real, the lexicographical ordering reduces to the real ordering, and

$$\cdots \leqslant \lambda_{n-1}^+ < \lambda_n^- \leqslant \lambda_n^+ < \lambda_{n+1}^- \leqslant \cdots.$$

Moreover, every eigenfunction of $\lambda_{2n}^\pm$ is 1-periodic while every eigenfunction of $\lambda_{2n+1}^\pm$ is 1-antiperiodic. By Floquet theory, the periodic eigenvalues characterize the spectrum of $L(\varphi)$ when considered as an operator on the real line with φ being periodically extended. In this case the spectrum is absolutely continuous and given by

$$\operatorname{spec}_{\mathbb{R}}(L(\varphi)) = \mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} (\lambda_n^-, \lambda_n^+)$$

The complementary and possibly empty intervals $(\lambda_n^-, \lambda_n^+)$ precisely describe the gaps in the continuous spectrum, hence one speaks of *spectral gaps*. One defines *gap lengths* γ_n and *mid points* τ_n by

$$\gamma_n = \lambda_n^+ - \lambda_n^- = \ell_n^2, \qquad \tau_n = \frac{\lambda_n^- + \lambda_n^+}{2}.$$

When some gap length vanishes, the corresponding gap is called *collapsed*, and otherwise *open*.

For a potential φ not of real type, the operator $L(\varphi)$ is not self-adjoint hence the periodic spectrum is complex in general. The gap lengths and mid points are defined as before but may be complex numbers. They are also no longer characterized by Floquet theory. Nevertheless, we speak of open and collapsed gaps.

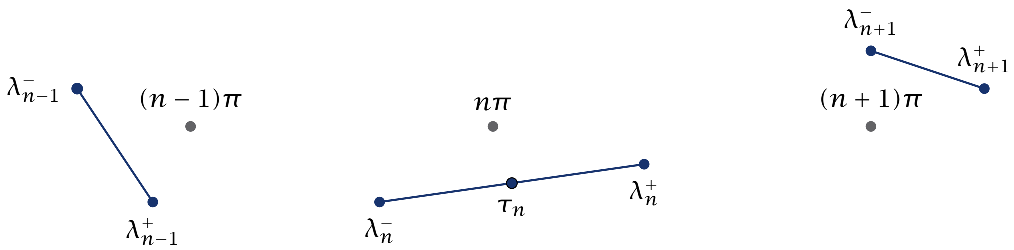

Figure 1.1.: Lexicographical ordering of the periodic eigenvalues.

The goal of this chapter is to extend the definition of the operator $L(\varphi)$ and its spectral theory to the case where φ is an element of the *Fourier Lebesgue space* FL_c^p, $1 \leqslant p < \infty$, motivated in the introduction. Even tough for any $p > 2$ there exists elements of FL_c^p which are not measures but more singular periodic distributions, the most important qualitative features of the spectrum persist under the transition to Fourier Lebesgue spaces.

To avoid distinguishing between periodic and anti-periodic functions we note that the periodic spectrum of $\varphi \in L_c^2$ coincides with the spectrum of the operator (1.1) considered on the interval $[0,2]$ with periodic boundary conditions, that is $L(\varphi)$ acting as an unbounded operator on $L_c^2(\mathbb{T}_2)$ with domain $H_c^1(\mathbb{T}_2)$. This motivates to define the periodic spectrum of $\varphi \in FL_c^p$ as the spectrum of the operator (1.1) acting as an unbounded operator on $FL_c^p(\mathbb{T}_2)$ with domain $FL_c^{1,p}(\mathbb{T}_2)$. Here φ is extended periodically to $[0,2]$ and viewed as an element of $FL_c^p(\mathbb{T}_2)$. In this setting, the operator $L(\varphi)$ is closed and its spectral theory appears as a natural extension of the case where φ is in L_c^2.

Theorem 1.1 *The periodic spectrum of any potential in FL_c^p, $1 < p < \infty$, is discrete and there exists a neighborhood U of the potential in FL_c^p and an integer $N > 0$ such that*

(i) For any $|n| > N$, each potential in U has precisely two periodic eigenvalues in the disc

$$D_n = \{\lambda \in \mathbb{C} : |\lambda - n\pi| < \pi/4\}.$$

The eigenvalues are 1-periodic for n even and 1-antiperiodic for n odd.

(ii) Every potential φ in U has exactly $4N + 2$ periodic eigenvalues in the box

$$B_N = \{\lambda \in \mathbb{C} : |\Re\lambda| < N\pi + \pi/2, \quad |\Im\lambda| \leqslant (1 + 8\|\varphi\|_p)^p\},$$

For N even, $2N + 2$ of them are 1-periodic and $2N$ of them are 1-antiperiodic while for N odd, $2N$ of them are 1-periodic and $2N + 2$ are 1-antiperiodic.

(iii) There are no other periodic eigenvalues.

(iv) If the potential is of real type, then all the eigenvalues are real. ⋊

In particular, the periodic spectrum of any $\varphi \in FL_c^p$ can be ordered lexicographically so that

$$\cdots \preccurlyeq \lambda_{n-1}^+ \preccurlyeq \lambda_n^- \preccurlyeq \lambda_n^+ \preccurlyeq \lambda_{n+1}^- \preccurlyeq \cdots, \qquad \lambda_n^\pm = n\pi + O(1),$$

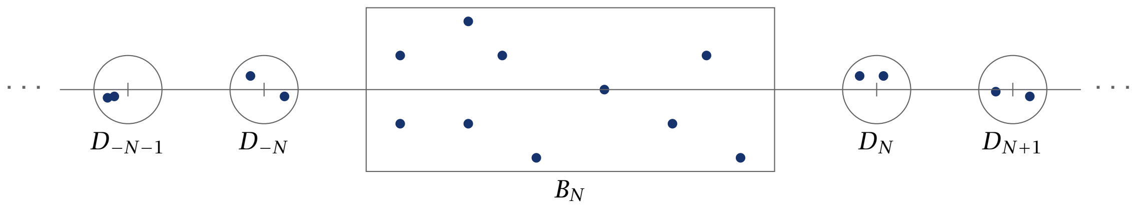

Figure 1.2.: Distribution of the periodic eigenvalues

Recall that $\lambda_n^\pm = n\pi + \ell_n^2$ for $\varphi \in L_c^2$. We obtain an analogous result for the case that φ is an element of the Fourier Lebesgue space FL_c^p.

Theorem 1.2 *For any potential φ in FL_c^p, $1 < p < \infty$, we have*

$$\lambda_n^\pm = n\pi + \ell_n^p$$

locally uniformly in φ. In more detail, there exists an absolute constant $C > 0$ so that

$$\sum_{n\in\mathbb{Z}} (|\lambda_n^+ - n\pi|^p + |\lambda_n^- - n\pi|^p) \leqslant C,$$

and this constant C can be chosen locally uniformly in φ. $\rtimes$

Due to the lexicographical ordering also the mid-points τ_n and the gap lengths γ_n are well defined.

Corollary 1.3 *For any potential φ in FL_c^p, $1 < p < \infty$, we have*

$$\tau_n = n\pi + \ell_n^p, \qquad \gamma_n = \ell_n^p,$$

locally uniformly in φ. $\rtimes$

We now turn to the Dirichlet and Neumann spectra of φ. Let us first recall their definition for the case that $\varphi \in L_c^2$. The Dirichlet and Neumann spectra are then most transparently defined in AKNS coordinates [2]. To this end, write $\varphi = (q + \mathrm{i}p, q - \mathrm{i}p)$ to obtain the AKNS representation of $L(\varphi)$,

$$L_{AKNS}(q,p) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} q & p \\ p & -q \end{pmatrix}.$$

The operators $L(\varphi)$ and $L(q,p)$ are unitarily equivalent,

$$L_{AKNS}(q,p) = T^* L(\varphi) T, \qquad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}.$$

When φ is of real type, then q and p can be chosen as real functions and $L_{AKNS}(q,p)$ is a real valued operator. Therefore, $L(\varphi)$ may be viewed as the complexification of $L_{AKNS}(q,p)$ when q and p are allowed to be complex valued. In AKNS coordinates, the Dirichlet and Neumann spectra of are defined with respect to the dense domains

$$\mathcal{A}_{dir} = \{ f = (f_-, f_+) \in H_c^1[0,1] : f_+(0) = 0 = f_+(1) \},$$
$$\mathcal{A}_{neu} = \{ g = (g_-, g_+) \in H_c^1[0,1] : g_-(0) = 0 = f_-(1) \}.$$

The transformation T maps those domains of $L_{AKNS}(q,p)$ to the following ones for $L(\varphi)$

$$\mathcal{D}_{dir} = \{f = (f_-, f_+) \in H_c^1[0,1] : (f_+ - f_-)\big|_0 = 0 = (f_+ - f_-)\big|_1\},$$
$$\mathcal{D}_{neu} = \{g = (g_-, g_+) \in H_c^1[0,1] : (g_+ + g_-)\big|_0 = 0 = (g_+ + g_-)\big|_1\}.$$

As is well known, the Dirichlet and Neumann eigenvalues are given as bi-infinite sequences $(\mu_n)_{n\in\mathbb{Z}}$ and $(\nu_n)_{n\in\mathbb{Z}}$ which can be ordered lexicographically so that

$$\cdots \preccurlyeq \mu_{n-1} \preccurlyeq \mu_n \preccurlyeq \mu_{n+1} \preccurlyeq \cdots, \qquad \mu_n = n\pi + \ell_n^2,$$
$$\cdots \preccurlyeq \nu_{n-1} \preccurlyeq \nu_n \preccurlyeq \nu_{n+1} \preccurlyeq \cdots, \qquad \nu_n = n\pi + \ell_n^2.$$

When φ is of real type, the Dirichlet and Neumann spectra are real and one has the relation

$$\lambda_n^- \leqslant \mu_n, \nu_n \leqslant \lambda_n^+.$$

The extension of the Dirichlet and Neumann domains of the operator $L(\varphi)$ to the case where φ is an element of the Fourier Lebesgue space FL_c^p is technically more involved than the extension of domain for the periodic spectrum. Note that the elements of the domains $\mathcal{D}_{dir}$ and $\mathcal{D}_{neu}$ are in $H^1[0,1]$ but in general not periodic, whereas every element in $FL_c^{1,p}$ is periodic. To avoid the introduction of a different notion of the Fourier Lebesgue spaces which allows to make sense of the Fourier Lebesgue regularity *locally* without enforcing periodic boundary conditions, we instead consider φ as a distribution defined on $[0,1]$ and construct an appropriate extension $\tilde{\varphi}$ to the interval $[0,2]$, which is 2-periodic, so that the periodic spectrum of the operator $L(\tilde{\varphi})$ coincides with the Dirichlet- and Neumann spectra for $\varphi \in L_c^2$. In this way, we define the Dirichlet and Neumann spectra for the case where φ is an element of FL_c^p – see Section 4 for details.

Theorem 1.4 *The Dirichlet and Neumann spectra of any potential in FL_c^p, $1 < p < \infty$, are discrete and there exists a neighborhood U of the potential in FL_c^p and an integer $N > 0$ such that*

(i) For any $|n| > N$, each $\varphi \in U$ has a simple Dirichlet respectively Neumann eigenvalue in the disc

$$D_n = \{\lambda \in \mathbb{C} : |\lambda - n\pi| < \pi/4\},$$

(ii) Each $\varphi \in U$ has exactly $2N+1$ Dirichlet respectively Neumann eigenvalues in the box

$$B_N = \{\lambda \in \mathbb{C} : |\Re\lambda| < N\pi + \pi/2, \quad |\Im\lambda| \leqslant (1 + 8\|\varphi\|_p)^p\},$$

(iii) There are no other Dirichlet or Neumann eigenvalues.

(iv) If the potential is of real type, then all Dirichlet and Neumann eigenvalues are real valued. ⋊

In particular, the Dirichlet and Neumann eigenvalues of any $\varphi \in FL_c^p$ can be ordered lexicographically so that

$$\cdots \preccurlyeq \mu_{n-1} \preccurlyeq \mu_n \preccurlyeq \mu_{n+1} \preccurlyeq \cdots, \qquad \mu_n = n\pi + O(1),$$
$$\cdots \preccurlyeq \nu_{n-1} \preccurlyeq \nu_n \preccurlyeq \nu_{n+1} \preccurlyeq \cdots, \qquad \nu_n = n\pi + O(1).$$

As in the case of the periodic spectrum, the asymptotic behavior of the Dirichlet and Neumann spectra actually reflects the asymptotic behavior of the Fourier coefficients of the potential.

Theorem 1.5 *For any potential φ in FL_c^p, $1 < p < \infty$,*

$$\mu_n(\varphi) = n\pi + \ell_n^p, \qquad \nu_n(\varphi) = n\pi + \ell_n^p,$$

locally uniformly in φ. ⋊

Finally, we introduce the notion of a *finite gap* potential, which is a potential φ with finitely many open gaps. That is, there exists an integer $N \geqslant 1$ so that $\gamma_n(\varphi) = 0$ for all $|n| > N$. Finite gap potentials are smooth, in fact real analytic, functions and the finite gap potentials of real type are known [37, 18] to be dense in any Sobolev space H_r^m, $m \geqslant 0$. The following is an immediate consequence.

Corollary 1.6 *For any $1 < p < \infty$, the finite gap potentials of real type are dense in FL_r^p.* ⋊

Related results. There exists a vast amount of literature on the spectral theory of the operator $L(\varphi)$ as well as the closely related Schrödinger operator $L(q) = -\partial_x^2 + q$ which appears in the Lax pair formulation of the Korteweg de Vries (KdV) equation

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u, \qquad x \in \mathbb{T}.$$

See e.g. the survey article of Djakov & Mityagin [18] for an overview. We only mention the results which are most closely related to this work. To the best of our knowledge there exists no record for the spectral theory of the operator $L(\varphi)$ in the Banach spaces FL_c^p. We therefore adapt the Hilbert space methods developed by Kappeler and collaborators [29, 30, 37] to the case $p \neq 2$ – see also Pöschel [63] for the case of the operator Schrödinger operator $L(q)$ with a potential q taken from the Fourier Lebesgue space. We note that from a spectral theoretical point of view, the case $q \in H^{-1}$ corresponds to the case $\varphi \in L_c^2$ – see e.g. Chodos [9] and Molnar & Widmer [58].

2. Fourier Lebesgue spaces

We define for $a > 0$, $s \in \mathbb{R}$, and $1 \leqslant p \leqslant \infty$ the *Fourier Lebesgue space*

$$FL^{s,p}(\mathbb{T}_a) = \{u \in S'(\mathbb{T}_a, \mathbb{C}) : (u_m) \in \ell^{s,p}(\mathbb{Z}), \quad u_m = [u, \mathrm{e}^{-\mathrm{i}2m\pi x/a}]_{\mathbb{T}_a}\},$$

where $\mathbb{T}_a = \mathbb{R}/a\mathbb{Z}$ and $[u, f]_{\mathbb{T}_a}$ denotes the linear action of the distribution u on a test function $f \in S(\mathbb{T}_a, \mathbb{C}) = \{\phi \in C^\infty(\mathbb{R}) : \phi(x + a) = \phi(x) \ \forall\ x \in \mathbb{R}\}$. For $u \in L^1_{\text{loc}}$ the action is given by

$$[u, f]_{\mathbb{T}_a} = \frac{1}{a}\int_{\mathbb{T}_a} u(x)f(x)\,\mathrm{d}x,$$

– see also Appendix A for a brief introduction into the theory of periodic distributions. The norm $\|u\|_{FL^{s,p}} \coloneqq \|(u_m)_{m\in\mathbb{Z}}\|_{\ell^{s,p}}$ makes $FL^{s,p}$ into a Banach space for any $s \in \mathbb{R}$ and any $1 \leqslant p \leqslant \infty$, which is reflexive for $1 < p < \infty$. Any element of $u \in FL^{s,p}(\mathbb{T}_a)$ can be represented by its Fourier Series

$$u = \sum_{m\in\mathbb{Z}} u_m \mathrm{e}_{2m/a}, \qquad \mathrm{e}_\alpha = \mathrm{e}^{\mathrm{i}\alpha\pi x},$$

which converges to u in the $\|\cdot\|_{FL^{s,p}}$-norm. For $1 \leqslant p < \infty$ the dual $(FL^{s,p}(\mathbb{T}_a))^*$ can be identified with $FL^{-s,p'}(\mathbb{T}_a)$ where p' denotes the exponent conjugated to p. For $u \in FL^{s,p}(\mathbb{T}_a)$ and $v \in FL^{-s,p'}(\mathbb{T}_a)$ one can define the *sesquilinear dual product*

$$\langle u, v\rangle_{FL^{s,p}(\mathbb{T}_a)\times FL^{-s,p'}(\mathbb{T}_a)} \coloneqq \sum_{m\in\mathbb{Z}} u_m\overline{v_m},$$

which is a natural extension of the $L^2(\mathbb{T}_a)$ inner product

$$\langle u, v\rangle_{L^2(\mathbb{T}_a)\times L^2(\mathbb{T}_a)} = \frac{1}{a}\int_{\mathbb{T}_a} u(x)\overline{v(x)}\,\mathrm{d}x.$$

We also define the *real dual product without complex conjugation*

$$[u, v]_{FL^{s,p}(\mathbb{T}_a)\times FL^{-s,p'}(\mathbb{T}_a)} \coloneqq \sum_{m\in\mathbb{Z}} u_m v_m,$$

which is a natural extension of the linear action of u on test functions $v \in S(\mathbb{T}_a, \mathbb{C})$. Clearly,

$$\langle u, v\rangle_{FL^{s,p}(\mathbb{T}_a)\times FL^{-s,p'}(\mathbb{T}_a)} = [u, \overline{v}]_{FL^{s,p}(\mathbb{T}_a)\times FL^{-s,p'}(\mathbb{T}_a)}.$$

Further, we introduce the space $FL_c^{s,p}(\mathbb{T}_a) \coloneqq FL^{s,p}(\mathbb{T}_a) \times FL^{s,p}(\mathbb{T}_a)$ with elements $f = (f_-, f_+)$. The Fourier series of any element f can be written by

$$f = \sum_{m\in\mathbb{Z}} (f_m^- e_{2m/a}^- + f_m^+ e_{2m/a}^+), \qquad e_\alpha^- \coloneqq (\mathrm{e}_{-\alpha}, 0), \quad e_\alpha^+ \coloneqq (0, \mathrm{e}_\alpha), \tag{1.2}$$

and with $\langle x\rangle \coloneqq 1 + |x|$ the norm on $FL_c^{s,p}(\mathbb{T}_a)$ is

$$\|f\|_{s,p} \coloneqq \left(\sum_{m\in\mathbb{Z}} \langle m\rangle^{sp}(|f_m^-|^p + |f_m^+|^p)\right)^{1/p}, \qquad 1 \leqslant p < \infty,$$
$$\|f\|_{s,\infty} \coloneqq \sup_{m\in\mathbb{Z}} \langle m\rangle^s (|f_m^-| + |f_m^+|).$$

The sesquilinear dual product for elements $f \in FL_c^{s,p}(\mathbb{T}_a)$ and $g \in FL_c^{-s,p'}(\mathbb{T}_a)$ is given by

$$\langle f, g\rangle \equiv \langle f, g\rangle_{FL_c^{s,p}\times FL_c^{-s,p'}} \coloneqq \langle f_-, g_-\rangle_{FL^{s,p}\times FL^{-s,p'}} + \langle f_+, g_+\rangle_{FL^{s,p}\times FL^{-s,p'}},$$

and we also define the real dual product

$$\langle f, g\rangle_r \coloneqq \langle f, \overline{g}\rangle.$$

Suppose F is an analytic functional on some neighborhood $U \subset FL_c^{s,p}$. Its differential is at any point $\varphi \in U$ a bounded linear functional $\mathrm{d}_\varphi F\colon FL_c^{s,p} \to \mathbb{C}$. By the Riesz Representation Theorem, it admits a uniquely determined L^2-*gradient*, which is an element of $FL_c^{-s,p'}$ and will be denoted by $\partial_\varphi F$, so that for any $h \in FL_c^{s,p}$

$$\mathrm{d}_\varphi F h = \langle \partial_\varphi F, h\rangle_r.$$

With this notion of the gradient, we endow the space $FL_c^{s,p}(\mathbb{T}_a)$ with the Poisson bracket

$$\{F, G\} \coloneqq -\mathrm{i}\langle \partial F, J\partial G\rangle_r, \qquad J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

which is defined whenever the L^2 gradients ∂F and ∂G are sufficiently regular to make sense of the dual product. If F is analytic on $FL_c^{-s,p'}(\mathbb{T}_a)$ and G is analytic on $FL_c^{s,p}(\mathbb{T}_a)$, then the regularity assumption is automatically satisfied.

To simplify notation we denote $FL_c^p(\mathbb{T}_a) \equiv FL_c^{0,p}(\mathbb{T}_a)$ and write $\|f\|_p \equiv \|f\|_{0,p}$. Moreover, the space of 1-periodic functions simply denoted by $FL_c^p \equiv FL_c^p(\mathbb{T}_1)$.

3. Periodic spectrum

In this section we investigate, given a potential $\varphi \in FL_c^p$, $1 \leqslant p < \infty$, the spectral theory of the operator

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}.$$

More to the point, we consider for any $\varphi \in FL_c^p(\mathbb{T}_2)$ the operator $L(\varphi)$ acting as an unbounded operator on $FL_c^p(\mathbb{T}_2)$ with domain $FL_c^{1,p}(\mathbb{T}_2)$. In this setting the operator $L(\varphi)$ is clearly densely defined and closed. Indeed, by decomposing $\varphi = \tilde{\varphi} + \varphi_\varepsilon$ where $\tilde{\varphi}$ is real analytic and $\|\varphi_\varepsilon\|_p \leqslant \varepsilon$ one shows that the operator $\Phi = \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}$ is $\frac{\mathrm{d}}{\mathrm{d}x}$-bounded with $\frac{\mathrm{d}}{\mathrm{d}x}$-bound ε. Therefore, $L(\varphi)$ is a relatively bounded perturbation of the closed operator $\frac{\mathrm{d}}{\mathrm{d}x}$ and hence closed – see [40, Chapter IV, § 1]. We proceed by showing that $L(\varphi)$ has a compact resolvent whence its spectrum is discrete and further obtain certain resolvent estimates allowing us to state a first rough localization of the periodic eigenvalues.

Proposition 3.1 *The periodic spectrum of any potential in $FL_c^p(\mathbb{T}_2)$, $1 \leqslant p < \infty$, is discrete and there exists a neighborhood U of the potential in $FL_c^p(\mathbb{T}_2)$ and an integer $N > 0$ such that*

(i) For any $|n| > N$ each potential in U has precisely two periodic eigenvalues in the disc

$$D_n = \{\lambda \in \mathbb{C} : |\lambda - n\pi| < \pi/4\}.$$

If, in addition, φ is 1-periodic, then the eigenvalues are 1-periodic for n even and 1-antiperiodic for n odd.

(ii) Every potential φ in U has exactly $4N + 2$ periodic eigenvalues in the box

$$B_N = \{\lambda \in \mathbb{C} : |\Re\lambda| < N\pi + \pi/2, \quad |\Im\lambda| \leqslant (1 + 8\|\varphi\|_p)^p\},$$

If, in addition, φ is 1-periodic, then for N even, $2N + 2$ of them are 1-periodic and $2N$ of them are 1-antiperiodic while for N odd, $2N$ of them are 1-periodic and $2N + 2$ are 1-antiperiodic.

(iii) There are no other periodic eigenvalues.

(iv) If the potential is of real type, then all the eigenvalues are real. ⋊

To obtain first bounds on the periodic spectrum of φ we analyze the resolvent $R_\varphi(\lambda) \coloneqq (\lambda - L(\varphi))^{-1}$ of $L(\varphi)$. Note that by the resolvent identity

$$R_\varphi(\lambda)(I - \Phi R_0(\lambda)) = R_0(\lambda).$$

Our goal is to invert $(I - \Phi R_0(\lambda))$ using a Neumann series to obtain bounds on the periodic spectrum of φ. By Young's inequality – see Lemma A.7

$$\|\Phi R_0(\lambda) f\|_p \leqslant \|\varphi\|_p \|R_0(\lambda) f\|_1. \tag{1.3}$$

We momentarily show that $\|R_0(\lambda) f\|_1 < 1/\|\varphi\|_p$ if the imaginary part of λ is large enough, giving a first rough localization of the spectrum. To give a precise statement, we introduce for any $n \in \mathbb{Z}$ and any $r > 0$ the vertical strips with a hole of radius r

$$\mathrm{Vert}_n(r) \coloneqq \{\lambda \in \mathbb{C} : |\Re\lambda - n\pi| \leqslant \pi/2, \quad |\lambda - n\pi| \geqslant r\}.$$

Lemma 3.2 *(i)* $R_0(\lambda)\colon FL^1_c(\mathbb{T}_2) \to FL^1_c(\mathbb{T}_2)$ *is compact for any* $\lambda \notin \pi\mathbb{Z}$.

(ii) For any $\lambda \in \mathbb{C}$ *and any* $1 \leqslant p < \infty$

$$\|R_0(\lambda)\|_{FL^p_c(\mathbb{T}_2)\to FL^1_c(\mathbb{T}_2)} \leqslant \frac{4p}{|\Im\lambda|^{1/p}} + \frac{1}{|\Im\lambda|}.$$

(iii) If $\lambda \in \mathrm{Vert}_n(r)$ *and* $0 < r \leqslant \pi/4$ *then for any* $1 \leqslant p < \infty$

$$\|R_0(\lambda)\|_{FL^p_c(\mathbb{T}_2)\to FL^1_c(\mathbb{T}_2)} \leqslant \frac{8p}{r}. \quad \rtimes$$

Proof. (i) For $\lambda \notin \pi\mathbb{Z}$ the operator R_0 is a well defined multiplication operator

$$R_0(\lambda)f = \sum_{m\in\mathbb{Z}} \frac{1}{\lambda - m\pi}(f^-_m e^-_m + f^+_m e^+_m).$$

Since $\frac{1}{|\lambda - m\pi|} \to 0$ for $m \to \pm\infty$, it is contained in the closure of finite rank operators on $FL^1_c(\mathbb{T}_2)$ and hence compact.

To proceed, write $R_0(\lambda)f = g$, then for $1 < p < \infty$

$$\|g_\pm\|_1 = \sum_{m\in\mathbb{Z}} \frac{|f^\pm_m|}{|\lambda - m\pi|} \leqslant \left(\sum_{m\in\mathbb{Z}} \frac{1}{|\lambda - m\pi|^{p'}}\right)^{1/p'} \|f_\pm\|_p,$$

with the usual modifications in the case $p = 1$. So it is to estimate the first factor in the cases (ii) and (iii).

(ii) Choose $n \in \mathbb{Z}$ such that $|\Re\lambda - n\pi| \leqslant \pi/2$, then for any $m \in \mathbb{Z}$

$$|\lambda - m\pi| \geqslant \frac{1}{\sqrt{2}}(|\Re\lambda - m\pi| + |\Im\lambda|) \geqslant |n - m| + \frac{|\Im\lambda|}{\sqrt{2}}.$$

By Lemma B.1 from the appendix $\sum_{m\neq n} \frac{1}{(\alpha+|m-n|)^{p'}} \leqslant 2p/\alpha^{p'-1}$ for any $\alpha > 0$. Therefore,

$$\left(\sum_{m\in\mathbb{Z}} \frac{1}{|\lambda - m\pi|^{p'}}\right)^{1/p'} \leqslant \left(\frac{2p}{|\Im\lambda/\sqrt{2}|^{p'-1}} + \frac{1}{|\Im\lambda|^{p'}}\right)^{1/p'} \leqslant \frac{4p}{|\Im\lambda|^{1/p}} + \frac{1}{|\Im\lambda|}.$$

On the other hand, in case $p = 1$ one checks $\sup_{m\in\mathbb{Z}} \frac{1}{|\lambda - m\pi|} \geqslant \sqrt{2}/|\Im\lambda|$.

(ii) Given $\lambda \in \mathrm{Vert}_n(r)$ with $0 < r \leqslant \pi/4$ we have for any $m \in \mathbb{Z}$

$$|\lambda - m\pi| \geqslant r/\sqrt{2} + |n - m|.$$

Similar to (i) one obtains for $1 < p < \infty$

$$\left(\sum_{m\in\mathbb{Z}} \frac{1}{|\lambda - m\pi|^{p'}}\right)^{1/p'} \leqslant \frac{4p}{r},$$

and for $p = 1$ one checks $\sup_{m\in\mathbb{Z}} \frac{1}{|\lambda - m\pi|} \geqslant 1/r$. ∎

Corollary 3.3 *The resolvent of* $\varphi \in FL^p_c(\mathbb{T}_2)$, $1 \leqslant p < \infty$, *is compact and depends analytically on* λ *on*

$$\left\{\lambda \in \mathbb{C} : \left(\frac{1}{|\Im\lambda|} + \frac{4p}{|\Im\lambda|^{1/p}}\right)\|\varphi\|_p < 1\right\}.$$

In particular, the periodic spectrum of φ *is discrete.* $\rtimes$

Proof. In view of the previous lemma $\|\Phi R_0(\lambda)\|_p \leqslant \left(\frac{1}{|\Im\lambda|} + \frac{4p}{|\Im\lambda|^{1/p}}\right)\|\varphi\|_p < 1$ provided that $|\Im\lambda|$ is sufficiently large. In this case R_φ can be written as a Neumann Series

$$R_\varphi(\lambda) = R_0(\lambda)(I - \Phi R_0(\lambda))^{-1},$$

and hence is analytic on $\{\lambda \in \mathbb{C} : \left(\frac{1}{|\Im\lambda|} + \frac{4p}{|\Im\lambda|^{1/p}}\right)\|\varphi\|_p < 1\}$. The compactness follows from the compactness of R_0. ∎

As a next step we show that $\|R_0\Phi R_0\|_{FL_c^p(\mathbb{T}_2)\to FL_c^1(\mathbb{T}_2)}$ is small for $|\lambda|$ sufficiently large which in view of the identity

$$(I - \Phi R_0(\lambda))^{-1} = (I + \Phi R_0)(I - \Phi R_0(\lambda)\Phi R_0(\lambda))^{-1}$$

allows us to determine the domain of analyticity of R_φ more accurately.

Lemma 3.4 *If $\varphi \in FL_c^p(\mathbb{T}_2)$ with $1 \leqslant p < \infty$, then for any $0 < r \leqslant \pi/4$ and any $\lambda \in \mathrm{Vert}_n(r)$*

$$\|R_0(\lambda)\Phi R_0(\lambda)\|_{FL_c^p(\mathbb{T}_2)\to FL_c^1(\mathbb{T}_2)} \leqslant \frac{c_p}{r^2}\left(\frac{\|\varphi\|_p}{|n|^{1/p}} + \|R_n\varphi\|_p\right),$$

where c_p is an absolute constant depending only on p. ⋊

Proof. Write $g = R_0\Phi R_0 f$. One checks by a straightforward computation that

$$g = \sum_{l,m\in\mathbb{Z}}\left(\frac{\varphi_{l+m}^- f_m^+}{(\lambda - l\pi)(\lambda - m\pi)}e_l^- + \frac{\varphi_{l+m}^+ f_m^-}{(\lambda - l\pi)(\lambda - m\pi)}e_l^+\right) = (g_-, g_+).$$

Since $\lambda \in \mathrm{Vert}_n(r)$, we find by a computation analogous to the previous lemma that for any $m \in \mathbb{Z}$

$$|\lambda - m\pi| \geqslant r/\sqrt{2} + |n - m|.$$

Therefore,

$$\|g_-\|_1 \leqslant 2\sum_{l,m\in\mathbb{Z}}\frac{|\varphi_{l+m}^-|\,|f_m^+|}{(r + |n-l|)(r + |n-m|)}.$$

We split this sum up into two parts $\Sigma_\flat + \Sigma_\sharp$ defined by the index sets

$$I_\flat = \{m, l \in \mathbb{Z} : |m - n| > |n|/2 \text{ or } |l - n| > |n|/2\},$$
$$I_\sharp = \{m, l \in \mathbb{Z} : |m - n| \leqslant |n|/2 \text{ and } |l - n| \leqslant |n|/2\}.$$

For $\Sigma_\flat$ we apply Hölder's inequality $(1/p + 1/p' = 1)$ and Lemma B.1 to obtain

$$\begin{aligned}\Sigma_\flat &\leqslant \left(\sum_{I_\flat}\frac{1}{(r + |n-l|)^{p'}(r + |n-m|)^{p'}}\right)^{1/p'}\left(\sum_{I_\flat}|\varphi_{l+m}^-|^p\,|f_m^+|^p\right)^{1/p}\\ &\leqslant \frac{c_p}{(r + |n|)^{1/p}}\|\varphi_-\|_p\|f_+\|_p.\end{aligned}$$

For $(m, l) \in I_\sharp$ we conclude $|m + l| \geqslant 2|n| - |m - n| - |l - n| \geqslant |n|$ so that

$$\begin{aligned}\Sigma_\sharp &\leqslant \left(\sum_{I_\sharp}\frac{1}{(r + |n-l|)^{p'}(r + |n-m|)^{p'}}\right)^{1/p'}\left(\sum_{I_\sharp}|\varphi_{l+m}^-|^p\,|f_m^+|^p\right)^{1/p}\\ &\leqslant \frac{c_p}{r^{2/p}}\|R_n\varphi_-\|_p\|f_+\|_p.\end{aligned}$$

With obvious modifications in the case $p = 1$. A similar estimate is obtained for $\|g_+\|_1$ which completes the proof. ∎

Corollary 3.5 *For any potential φ in $FL^p_c(\mathbb{T}_2)$, $1 \leqslant p < \infty$, there exists a neighborhood U_φ and an integer $N_\varphi \geqslant 0$ such that the resolvent of any potential in U_φ is compact and analytic in λ on $\mathbb{C} \setminus \left(B_{N_\varphi} \cup \bigcup_{|n|>N_\varphi} D_n\right)$, where*

$$
\begin{aligned}
B_N &= \{\lambda \in \mathbb{C} : |\Re\lambda| < N\pi + \pi/2, \quad |\Im\lambda| \leqslant N\},\\
D_n &= \{\lambda \in \mathbb{C} : |\lambda - n\pi| < \pi/4\}.
\end{aligned}
$$

Proof. First, choose N_φ according to Lemma 3.4 such that

$$
\|\Phi R_0(\lambda)\Phi R_0(\lambda)\|_{FL^p_c(\mathbb{T}_2)\to FL^p_c(\mathbb{T}_2)} \leqslant \|\varphi\|_p \|R_0(\lambda)\Phi R_0(\lambda)\|_{FL^p_c(\mathbb{T}_2)\to FL^1_c(\mathbb{T}_2)} \leqslant 1/4
$$

for all $\lambda \in \mathrm{Vert}_n(\pi/4)$ and $|n| > N_\varphi$. Furthermore, Lemma 3.4 clearly shows that one can choose a neighborhood U_φ of φ such that $\|\Phi R_0(\lambda)\Phi R_0(\lambda)\|_{FL^p_c(\mathbb{T}_2)\to FL^p_c(\mathbb{T}_2)} \leqslant 1/2$ holds for any potential in U_φ and for all $\lambda \in \mathrm{Vert}_n(\pi/4)$ with $|n| > N_\varphi$. Then R_ψ is compact and analytic in λ on $\bigcup_{|n|>N_\varphi} \mathrm{Vert}_n(\pi/4)$ for all $\psi \in U_\varphi$ and we conclude after possibly increasing N_φ according to Corollary 3.3. ∎

Since the straight line $[0, \varphi]$ is compact in $FL^p_c(\mathbb{T}_2)$, we may always assume that U_φ is connected and contains the zero potential. We proceed by defining the Cauchy-Riesz projectors

$$
P_N(\psi) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{N_\varphi}} R_\psi(\lambda)\,\mathrm{d}\lambda, \qquad P_n(\psi) = \frac{1}{2\pi\mathrm{i}} \int_{\partial D_n} R_\psi(\lambda)\,\mathrm{d}\lambda, \quad |n| > N_\varphi, \tag{1.4}
$$

which are analytic on U_φ.

To distinguish between 1-periodic and 1-antiperiodic eigenfunctions, we introduce the closed subspaces of FL^p_c

$$
\begin{aligned}
FL^p_{per+}(\mathbb{T}_2) &\coloneqq \{f \in FL^p_c(\mathbb{T}_2) : f^+_{2m+1} = f^-_{2m+1} = 0 \text{ for all } m \in \mathbb{Z}\},\\
FL^p_{per-}(\mathbb{T}_2) &\coloneqq \{f \in FL^p_c(\mathbb{T}_2) : f^+_{2m} = f^-_{2m} = 0 \text{ for all } m \in \mathbb{Z}\},
\end{aligned}
$$

and note that $FL^p_{per+}(\mathbb{T}_2) \oplus FL^p_{per-}(\mathbb{T}_2) = FL^p_c(\mathbb{T}_2)$. Moreover, $FL^p_c(\mathbb{T}_1)$ can be canonically identified with $FL^p_{per+}(\mathbb{T}_2)$.

Lemma 3.6 *Suppose $\varphi \in FL^p_c$, then $L(\varphi)$ leaves the spaces $FL^p_{per+}(\mathbb{T}_2)$ and $FL^p_{per-}(\mathbb{T}_2)$ invariant.*

Proof. Write $L(\varphi) = R\partial_x + \Phi$ with $R = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix}$. The operator $R\partial_x$ is a Fourier multiplier and thus leaves $FL^p_{per\pm}(\mathbb{T}_2)$ invariant. Since φ is 1-periodic by assumption, $\varphi \in FL^p_{per+}(\mathbb{T}_2)$ and one easily verifies that $\Phi f \in FL^p_{per\pm}$ given $f \in FL^{1,p}_c \cap FL^p_{per\pm}$. ∎

Consequently, for φ is 1-periodic, the projectors

$$
\begin{aligned}
P^\pm_N(\varphi) &\coloneqq P_N(\varphi)\big|_{FL^p_{per\pm}} : FL^p_{per\pm} \to FL^p_{per\pm},\\
P^\pm_n(\varphi) &\coloneqq P_n(\varphi)\big|_{FL^p_{per\pm}} : FL^p_{per\pm} \to FL^p_{per\pm}, \quad |n| > N_\varphi,
\end{aligned}
\tag{1.5}
$$

are analytic on an open neighborhood of φ within $FL^p_c(\mathbb{T}_1)$.

Proof of Proposition 1.1. (i): Since P_n for $|n| > N_\varphi$ is analytic on U_φ, the dimension of its range $\mathcal{P}_n$ is constant on U_φ. In particular,

$$\dim \mathcal{P}_n(\psi) = \dim \mathcal{P}_n(0) = 2, \qquad \psi \in U_\varphi,$$

which proves that D_n for $|n| > N_\varphi$ contains precisely two periodic eigenvalues. Moreover, if φ is actually 1-periodic, then applying the same deformation argument to the projectors $P_n^\pm$ defined in (1.5) shows that the two eigenvalues of φ are 1-periodic if n is even or 1-antiperiodic if n is odd.

(ii): The same reasoning applied to P_N with $N \equiv N_\varphi$ shows

$$\dim \mathcal{P}_N(\psi) = \dim \mathcal{P}_N(0) = 4N + 2, \qquad \psi \in U_\varphi.$$

Therefore, B_N contains precisely $4N+2$ periodic eigenvalues. If, in addition φ is 1-periodic, then applying the deformation argument to the projectors $P_N^\pm$ shows that for N even, $2N+2$ are 1-periodic and $2N$ are 1-antiperiodic, while for N odd, $2N$ are 1-periodic and $2N+2$ are 1-antiperiodic.

(iii): Outside of the set $B_N \cup \bigcup_{|n|>N_\varphi} D_n$ the resolvent is analytic, hence there are no other periodic eigenvalues.

(iv): Suppose φ is of real type, that is $\overline{\varphi_\pm} = \varphi_\mp$ then for any $f \in \mathcal{D}_{per} \subset FL_c^{1,p}(\mathbb{T}_2) \hookrightarrow FL_c^{p'}(\mathbb{T}_2)$ by Lemma A.8

$$\langle \Phi f, f \rangle_{\mathbb{T}_2} = \langle \varphi_- f_+, f_- \rangle_{\mathbb{T}_2} + \langle \varphi_+ f_-, f_+ \rangle_{\mathbb{T}_2} = \langle f, \Phi f \rangle_{\mathbb{T}_2}.$$

Consequently, any periodic eigenvalue of φ is real. ∎

Lemma 3.7 *Suppose $\varphi \in FL_c^p$, $1 \leqslant p < \infty$, then for each $|n| > N_\varphi$ the functions*

$$U_\varphi \to \mathbb{C}, \quad \varphi \mapsto \tau_n, \qquad U_\varphi \to \mathbb{C}, \quad \varphi \mapsto \frac{\gamma_n^2}{2},$$

are analytic. ⋊

Proof. The Cauchy-Riesz P_n, $|n| > N_\varphi$, projectors defined in (1.4) are analytic on U_φ and their range $\mathcal{P}_n$ has dimension two. More to the point, $\mathcal{P}_n$ is spanned by the two (generalized) eigenfunctions corresponding to the eigenvalues λ_n^- and λ_n^+, hence one has for any $\psi \in U_\varphi$,

$$\tau_n(\psi) = \frac{1}{2}\operatorname{Tr}\Big(L(\psi)\big|_{\mathcal{P}_n(\psi)}\Big), \qquad \frac{\gamma_n^2(\psi)}{2} = \operatorname{Tr}\Big((L(\psi) - \tau_n(\psi)\mathrm{Id}_{\mathcal{P}_n(\psi)})^2\big|_{\mathcal{P}_n(\psi)}\Big).$$

After possibly shrinking U_φ, the ranges $\mathcal{P}_n(\varphi)$ and $\mathcal{P}_n(\psi)$ are isomorphic for all $\psi \in U_\varphi$ and the isomorphism $T_\psi \colon \mathcal{P}_n(\varphi) \to \mathcal{P}_n(\psi)$ depends analytically on ψ – see [40, Remark 4.4]. Let $A(\psi) = T_\psi^{-1} L(\psi) T_\psi$ then

$$\tau_n(\psi) = \frac{1}{2}\operatorname{Tr}\Big(A(\psi)\big|_{\mathcal{P}_n(\varphi)}\Big), \qquad \frac{\gamma_n^2(\psi)}{2} = \operatorname{Tr}\Big((A(\psi) - \tau_n(\psi)\mathrm{Id}_{\mathcal{P}_n(\varphi)})^2\big|_{\mathcal{P}_n(\varphi)}\Big),$$

and the analyticity follows immediately. ∎

4. Dirichlet and Neumann spectra

In the classical setting where φ is an element of L_c^2, the Dirichlet- and Neumann spectra are defined as the spectra of the operator (1.1)

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}$$

acting as an unbounded operator on $L^2[0,1]$ with domains

$$\begin{aligned} \mathcal{D}_{dir} &= \{ f \in H_c^1[0,1] : (f_- - f_+)|_0 = 0 = (f_- - f_+)|_1 \}, \\ \mathcal{D}_{neu} &= \{ g \in H_c^1[0,1] : (g_- + g_+)|_0 = 0 = (g_- + g_+)|_1 \}, \end{aligned} \tag{1.6}$$

respectively. In this section we derive an equivalent definition of the spectrum which has a natural extension to the case where φ is in the Fourier-Lebesgue space FL^p, $1 < p < \infty$. The main issue with the classical definition is that one has to be able to make sense of the regularity of a function f *locally* without implying any boundary conditions – here in the sense that $f \in L_c^2 \cap H_{\mathrm{loc}}^1$. However, the condition that f is an element of the space $FL_c^{1,p}(\mathbb{T}_a)$ is a *nonlocal* property forcing f to be a-periodic. Recall that in the case of the periodic spectrum one considers instead of $L(\varphi)$ acting on $L_c^2[0,1]$ with domain

$$\mathcal{D}_{per} = \{ f \in L_c^2[0,1] : f \in H_{\mathrm{loc}}^1 \text{ and } f(0) = \pm f(1) \},$$

the operator $L(\varphi)$ acting on $L_c^2(\mathbb{T}_2)$ with domain

$$\tilde{\mathcal{D}}_{per} = H_c^1(\mathbb{T}_2).$$

The spectra of the two operators coincide but the latter definition does not require to make sense of the regularity of f locally and has a natural extension to the Fourier-Lebesgue space.

To derive a similar procedure for the Dirichlet- and Neumann spectra we first reconsider the AKNS representation of $L(\varphi)$ given by

$$L_{AKNS}(q,p) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} q & p \\ p & -q \end{pmatrix},$$

with dense domains

$$\begin{aligned} \mathcal{A}_{dir} &= \{ g \in H_c^1[0,1] : g_2(0) = 0 = g_2(1) \}, \\ \mathcal{A}_{neu} &= \{ g \in H_c^1[0,1] : g_1(0) = 0 = g_1(1) \}. \end{aligned}$$

The condition that $g \in \mathcal{A}_{dir}$ solves $L_{AKNS}(q,p)g = \lambda g$ is equivalent to the system

$$\begin{pmatrix} -g_2' \\ g_1' \end{pmatrix} = \begin{pmatrix} \lambda g_1 - q g_1 - p g_2 \\ \lambda g_2 - p g_1 + q g_2 \end{pmatrix}. \tag{1.7}$$

Since g_2 vanishes on $\partial[0,1]$ by assumation, it admits an odd extension to $[0,2]$

$$g_2^{odd}(x) = \begin{cases} g_2(x), & 0 \leqslant x \leqslant 1, \\ -g_2(2-x), & 1 \leqslant x \leqslant 2, \end{cases}$$

which is an element of $H^1(\mathbb{T}_2, \mathbb{C})$. Equation (1.7) further suggests to consider the even extension of g_1 to $[0,2]$

$$g_1^{even}(x) = \begin{cases} g_1(x), & 0 \leqslant x \leqslant 1, \\ g_1(2-x), & 1 \leqslant x \leqslant 2, \end{cases}$$

which always exists as an element of $H^1(\mathbb{T}_2,\mathbb{C})$. Similarly, the extension of p to $[0,2]$ has to be odd and the extension of q to $[0,2]$ has to be even – both extensions are in $L^2_c(\mathbb{T}_2)$. One then easily checks that given $g \in \mathcal{A}_{dir}$ solving $L_{AKNS}(q,p)g = \lambda g$ its extension (g_1^{even}, g_2^{odd}) is an element of $H^1_c(\mathbb{T}_2)$ which solves

$$L_{AKNS}(q^{even}, p^{odd})(g_1^{even}, g_2^{odd}) = \lambda(g_1^{even}, g_2^{odd}).$$

A similar consideration for an element $h \in \mathcal{A}_{neu}$ solving $L_{AKNS}(q,p)h = \lambda h$ shows that (h_1^{odd}, h_2^{even}) is an element of $H^1_c(\mathbb{T}_2)$ which solves

$$L_{AKNS}(q^{even}, p^{odd})(h_1^{odd}, h_2^{even}) = \lambda(h_1^{odd}, h_2^{even}).$$

Recall that $L(\varphi)$ and $L_{AKNS}(q,p)$ are unitarily equivalent,

$$L_{AKNS}(q,p) = T^{-1}L(\varphi)T, \quad \varphi = T(q,p), \qquad T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}, \quad T^* = T^* = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}.$$

Suppose $(g_1, g_2) = T^{-1}f = \frac{1}{\sqrt{2}}(f_+ + f_-, \mathrm{i}f_+ - \mathrm{i}f_-)$, then

$$\begin{aligned} T(g_1^{even}, g_2^{odd}) &= \frac{1}{2}\begin{pmatrix} (f_- + f_+)^{even} + (f_- - f_+)^{odd} \\ (f_- + f_+)^{even} - (f_- - f_+)^{odd} \end{pmatrix} \\ &= \begin{cases} f(x), & 0 \leqslant x \leqslant 1, \\ \check{f}(2-x), & 1 \leqslant x \leqslant 2, \end{cases} \qquad \check{f} = (f_+, f_-), \\ &=: f^{dir}. \end{aligned}$$

Similarly for $(h_1, h_2) = T^{-1}f = \frac{1}{\sqrt{2}}(f_+ + f_-, \mathrm{i}f_+ - \mathrm{i}f_-)$, we find

$$\begin{aligned} T(h_1^{odd}, h_2^{even}) &= \frac{1}{2}\begin{pmatrix} (f_- + f_+)^{odd} + (f_- - f_+)^{even} \\ (f_- + f_+)^{odd} - (f_- - f_+)^{even} \end{pmatrix} \\ &= \begin{cases} f(x), & 0 \leqslant x \leqslant 1, \\ -\check{f}(2-x), & 1 \leqslant x \leqslant 2, \end{cases} \qquad \check{f} = (f_+, f_-), \\ &=: f^{neu}. \end{aligned}$$

We thus arrive at the following conclusion.

Lemma 4.1 *(i) Suppose $f \in \mathcal{D}_{dir}$ solves $L(\varphi)f = \lambda f$ for some $\lambda \in \mathbb{C}$, then f^{dir} is an element of $H^1_c(\mathbb{T}_2)$ which solves $L(\varphi^{dir})f^{dir} = \lambda f^{dir}$.*

(ii) Suppose $f \in \mathcal{D}_{neu}$ solves $L(\varphi)f = \lambda f$ for some $\lambda \in \mathbb{C}$, then f^{neu} is an element of $H^1_c(\mathbb{T}_2)$ which solves $L(\varphi^{dir})f^{neu} = \lambda f^{neu}$. ⋊

Consequently, every Dirichlet eigenvalue μ_n of $\varphi \in L^2$ is actually a periodic eigenvalue of the potential φ^{dir}. The same is true for any Neumann eigenvalue ν_n. It follows from a simple counting argument that these are *all* the periodic eigenvalues of φ^{dir}. Therefore, the spectral problem of the Dirichlet and Neumann eigenvalues of φ is reduced to the spectral problem of the periodic eigenvalues of φ^{dir}. In the sequel we use this reduction to extend the definition of Dirichlet and Neumann eigenvalues to the case where φ is in the Fourier Lebesgue space FL^p_c.

We begin by deriving a Fourier representation of the extensions f^{dir} and f^{neu} for elements f of $H^1[0,1]$. A straightforward computation gives

$$\begin{aligned}\langle f^{dir}, e_m^+\rangle_{\mathbb{T}_2} &= \frac{1}{2}\int_0^1 f_+(x)\overline{\mathrm{e}_m(x)}\,\mathrm{d}x + \frac{1}{2}\int_1^2 f_-(2-x)\overline{\mathrm{e}_m(x)}\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^1 f_+(x)\overline{\mathrm{e}_m(x)}\,\mathrm{d}x + \frac{1}{2}\int_0^1 f_-(y)\overline{\mathrm{e}_{-m}(y)}\,\mathrm{d}x,\end{aligned}$$

and similarly

$$\begin{aligned}\langle f^{dir}, e_m^-\rangle_{\mathbb{T}_2} &= \frac{1}{2}\int_0^1 f_-(x)\overline{\mathrm{e}_{-m}(x)}\,\mathrm{d}x + \frac{1}{2}\int_1^2 f_+(2-x)\overline{\mathrm{e}_{-m}(x)}\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^1 f_-(x)\overline{\mathrm{e}_{-m}(x)}\,\mathrm{d}x + \frac{1}{2}\int_0^1 f_+(y)\overline{\mathrm{e}_m(y)}\,\mathrm{d}x,\end{aligned}$$

so that we arrive at

$$\begin{aligned}f^{dir} &= \sum_{m\in\mathbb{Z}} \frac{1}{2}(\langle f_+, \mathrm{e}_m\rangle_{\mathbb{T}_1} + \langle f_-, \mathrm{e}_{-m}\rangle_{\mathbb{T}_1})e_m^+ + \frac{1}{2}(\langle f_-, \mathrm{e}_{-m}\rangle_{\mathbb{T}_1} + \langle f_+, \mathrm{e}_m\rangle_{\mathbb{T}_1})e_m^-\\ &= \sum_{m\in\mathbb{Z}} \frac{1}{2}\langle f, E_m^{dir}\rangle_{\mathbb{T}_1} E_m^{dir}, \qquad E_m^{dir} \coloneqq e_m^+ + e_m^-. \end{aligned} \tag{1.8}$$

And in analogous fashion

$$f^{neu} = \sum_{m\in\mathbb{Z}} \frac{1}{2}\langle f, E_m^{neu}\rangle_{\mathbb{T}_1} E_m^{neu}, \qquad E_m^{neu} \coloneqq e_m^+ - e_m^-. \tag{1.9}$$

Therefore, we introduce the following subspaces of $FL_c^{s,p}(\mathbb{T}_2)$

$$\begin{aligned}FL_{dir}^{s,p}(\mathbb{T}_2) &= \Big\{f = \sum_{m\in\mathbb{Z}} a_m E_m^{dir} \,:\, (a_m) \in \ell^{s,p}\Big\},\\ FL_{neu}^{s,p}(\mathbb{T}_2) &= \Big\{f = \sum_{m\in\mathbb{Z}} b_m E_m^{neu} \,:\, (b_m) \in \ell^{s,p}\Big\}.\end{aligned}$$

Both are closed subspaces of $FL_c^{s,p}(\mathbb{T}_2)$. Furthermore, their sum is direct and equals the whole space

$$FL_c^{s,p}(\mathbb{T}_2) = FL_{dir}^{s,p}(\mathbb{T}_2) \oplus FL_{neu}^{s,p}(\mathbb{T}_2).$$

We now show that they are natural extensions of the domains $\mathcal{D}_{dir}$ and $\mathcal{D}_{neu}$.

Lemma 4.2 *The mappings*

$$\mathcal{D}_{dir} \to FL_{dir}^{1,2}(\mathbb{T}_2), \quad f \mapsto f^{dir}, \qquad \mathcal{D}_{neu} \to FL_{neu}^{1,2}(\mathbb{T}_2), \quad f \mapsto f^{neu},$$

are isomorphisms. ⋊

Proof. Recall that $\mathcal{D}_{dir} = T(\mathcal{A}_{dir})$ where $\mathcal{A}_{dir} = H^1[0,1] \times H_0^1[0,1]$. Let

$$\begin{aligned}H_{\sin}^1(\mathbb{T}_2) &\coloneqq \{f = \sum_{m\in\mathbb{Z}} a_m \sin(m\pi x) \,:\, a_m + a_{-m} = 0\},\\ H_{\cos}^1(\mathbb{T}_2) &\coloneqq \{f = \sum_{m\in\mathbb{Z}} b_m \cos(m\pi x) \,:\, b_m - b_{-m} = 0\},\end{aligned}$$

and note that the restriction maps

$$\begin{aligned}R_{\sin}&\colon H_{\sin}^1(\mathbb{T}_2) \to H_0^1[0,1], \quad f \mapsto f|_{[0,1]},\\ R_{\cos}&\colon H_{\cos}^1(\mathbb{T}_2) \to H^1[0,1], \quad f \mapsto f|_{[0,1]},\end{aligned}$$

are isomorphisms – see e.g. [54] –, hence

$$H^1_{\cos}(\mathbb{T}_2) \times H^1_{\sin} = (R_{\cos} \times R_{\sin})^{-1} T^{-1}(\mathcal{D}_{dir}).$$

Finally, note that $FL^{1,2}_{dir}(\mathbb{T}_2)$ is isomorphic to $T(H^1_{\cos}(\mathbb{T}_2) \times H^1_{\sin})$. Indeed, for any element

$$g = \frac{1}{2} \sum_{m\in\mathbb{Z}} (b_m(\mathrm{e}_m + \mathrm{e}_{-m}), -\mathrm{i}a_m(\mathrm{e}_m - \mathrm{e}_{-m})) \in H^1_{\cos}(\mathbb{T}_2) \times H^1_{\sin}$$

we find

$$\begin{aligned} Tg &= \frac{1}{\sqrt{8}} \sum_{m\in\mathbb{Z}} ((b_m - a_m)\mathrm{e}_{-m} + (b_m + a_m)\mathrm{e}_m, (b_m - a_m)\mathrm{e}_m + (b_m + a_m)\mathrm{e}_{-m}) \\ &= \frac{1}{\sqrt{2}} \sum_{m\in\mathbb{Z}} (b_m - a_m) E^{dir}_m, \end{aligned}$$

using that $a_m + a_{-m} = 0$ and $b_m - b_{-m} = 0$. Conversely, if $f = \sum_{m\in\mathbb{Z}} f^{dir}_m E^{dir}_m$, then

$$T^{-1} f = \frac{1}{\sqrt{2}} \sum_{m\in\mathbb{Z}} \Big((f^{dir}_m + f^{dir}_{-m}) \cos(m\pi x), (-f^{dir}_m + f^{dir}_{-m}) \sin(m\pi x) \Big).$$

This establishes the isomorphism property. The proof for $\mathcal{D}_{neu}$ is analogous. ∎

We also have to make sure that given $\varphi \in FL^p_c$ the extension φ^{dir} is actually an element of $FL^p_c(\mathbb{T}_2)$ to use the spectral theory of the periodic case. Indeed, if φ is a smooth 1-periodic function then, in contrast to the case of elements of $\mathcal{D}_{dir}$ or $\mathcal{D}_{neu}$, one cannot expect φ^{dir} to be an element of $FL^{1,p}_c(\mathbb{T}_2)$ since the extension may destroy regularity. It turns out, however, that φ^{dir} will be at least an element of $FL^p_c(\mathbb{T}_2)$ which is enough for our purposes.

Lemma 4.3 *For any* $1 < p < \infty$ *the mappings*

$$FL^p_c(\mathbb{T}_1) \to FL^p_{dir}(\mathbb{T}_2), \quad f \mapsto f^{dir}, \qquad FL^p_c(\mathbb{T}_1) \to FL^p_{neu}(\mathbb{T}_2), \quad f \mapsto f^{neu},$$

are bounded and hence analytic. ⋊

Proof. Any $f \in FL^p_c(\mathbb{T}_1)$ has the form $f = \sum_{m\in\mathbb{Z}} (f^-_{2m} e^-_{2m} + f^+_{2m} e^+_{2m})$ with

$$\|f\|_{FL^p_c(\mathbb{T}_1)} = \left(\sum_{m\in\mathbb{Z}} (|f^-_{2m}|^p + |f^+_{2m}|^p) \right)^{1/p} < \infty.$$

Write $f^{dir}_m = \frac{1}{2}(\langle f_+, \mathrm{e}_m\rangle_{\mathbb{T}_1} + \langle f_-, \mathrm{e}_{-m}\rangle_{\mathbb{T}_1})$ so that $f^{dir} = \sum_{m\in\mathbb{Z}} f^{dir}_m E^{dir}_m$. It is to show that the ℓ^p-norm of the sequence (f^{dir}_m) can be bounded by $\|f\|_{FL^1_c(\mathbb{T}_1)}$. Clearly, one has $f^{dir}_{2l} = f^+_{2l} + f^-_{2l}$ while

$$\begin{aligned} f^{dir}_{2l+1} &= \sum_{k\in\mathbb{Z}} (f^+_{2k}\langle \mathrm{e}_{2k}, \mathrm{e}_{2l+1}\rangle + f^-_{2k}\langle \mathrm{e}_{-2k}, \mathrm{e}_{-2l-1}\rangle) \\ &= \sum_{k\in\mathbb{Z}} \frac{2\mathrm{i}}{2k - 2l - 1} (f^+_{2k} - f^-_{2k}). \end{aligned}$$

Consequently, the map $(f^+_{2m})_{m\in\mathbb{Z}} \mapsto (f^{dir}_m)_{m\in\mathbb{Z}}$ is the sum of a projection and a Hilbert transform and thus bounded as a map from $\ell^p \to \ell^p$ for any $1 < p < \infty$ – see Lemma C.1. ∎

For any $\varphi \in FL^p_c$, $1 < p < \infty$, we may thus consider the operator $L(\varphi^{dir})$ acting as an unbounded operator on $FL^p_c(\mathbb{T}_2)$ with domain $FL^{1,p}_c(\mathbb{T}_2)$. For this operator Proposition 3.1 applies showing that its spectrum is discrete and can be localized locally uniformly in φ. Moreover, in the classical case $p = 2$ it follows from the preceding discussion that each eigenvalue of $L(\varphi^{dir})$ is either a Dirichlet- or a Neumann-eigenvalue of φ. The following result extends this point of view to the Fourier Lebesgue spaces.

Lemma 4.4 *For any $1 < p < \infty$ the operator $L(\varphi^{dir})$ leaves the subspaces FL^p_{dir} and FL^p_{neu} invariant. In more detail,*

$$L(\varphi^{dir})\colon FL^{1,p}_{dir} \to FL^p_{dir}, \qquad L(\varphi^{dir})\colon FL^{1,p}_{neu} \to FL^p_{neu},$$

are bounded operators. ⋊

Proof. Write $L = R\partial_x + \Phi$. Clearly, for any $m \in \mathbb{Z}$,

$$R\partial_x E^{dir}_m = (m\pi)E^{dir}_m, \qquad R\partial_x E^{neu}_m = (m\pi)E^{neu}_m,$$

while for φ^{dir} we easily check that

$$\Phi E^{dir}_m = \sum_k \varphi^{dir}_k (\mathrm{e}_{-k}\mathrm{e}_m, \mathrm{e}_k\mathrm{e}_{-m}) = \sum_k \varphi^{dir}_k E^{dir}_{k-m}$$

$$\Phi E^{neu}_m = \sum_k \varphi^{dir}_k (-\mathrm{e}_{-k}\mathrm{e}_m, \mathrm{e}_k\mathrm{e}_{-m}) = -\sum_k \varphi^{dir}_k E^{neu}_{k-m}.$$

The claimed invariance now follows with Young's inequality – see Lemma A.8. ∎

We define the *Dirichlet spectrum of* $\varphi \in FL^p_c$ to be the spectrum of the operator $L(\varphi^{dir})|_{FL^{1,p}_{dir}}$, and we define the *Neumann spectrum of* $\varphi \in FL^p_c$ to be the spectrum of the operator $L(\varphi^{dir})|_{FL^{1,p}_{neu}}$. We now prove Theorem 1.4 with this definition.

Proof of Theorem 1.4. Consider the operator $L(\varphi^{dir})$ acting on $FL^p_c(\mathbb{T}_2)$ with domain $FL^{1,p}_c(\mathbb{T}_2)$. By the analysis of Section 3 and Lemma 4.3 the spectrum of this operator is discrete and one can choose an neighborhood U_φ of φ in FL^p_c and an integer $N_\varphi \geqslant 0$ so that the resolvent is analytic in λ on $\mathbb{C} \setminus (B_{N_\varphi} \cup \bigcup_{|n|>N_\varphi} D_n)$ for any $\psi \in U_\varphi$. Without loss one may assume that U_φ is connected and contains the origin. Moreover, the Cauchy-Riesz projectors $P_{N_\varphi}(\psi)$ and $P_n(\psi)$ for $|n| > N_\varphi$ defined in (1.4) are analytic on U_φ. In particular, the dimension of their range is constant on U_φ. Moreover, by Lemma 4.4 they leave the spaces FL^p_{dir} and FL^p_{neu} invariant. We thus define

$$P^{dir}_N(\varphi) \coloneqq P_N(\varphi)|_{FL^{1,p}_{dir}}, \qquad P^{dir}_n(\varphi) \coloneqq P_n(\varphi)|_{FL^{1,p}_{dir}}, \quad |n| > N_\varphi,$$

$$P^{neu}_N(\varphi) \coloneqq P_N(\varphi)|_{FL^{1,p}_{neu}}, \qquad P^{neu}_n(\varphi) \coloneqq P_n(\varphi)|_{FL^{1,p}_{neu}}, \quad |n| > N_\varphi.$$

One easily checks that for the zero potential

$$\dim P^{dir}_N(0) = 2N+1, \qquad \dim P^{dir}_n(0) = 1, \quad |n| > N_\varphi,$$

$$\dim P^{neu}_N(0) = 2N+1, \qquad \dim P^{neu}_n(0) = 1, \quad |n| > N_\varphi,$$

which shows that D_n for $|n| > N_\varphi$ contains precisely one Dirichlet- and one Neumann-eigenvalue, while B_N contains precisely $2N+1$ Dirichlet- and $2N+1$ Neumann-eigenvalues when counted with multiplicities. This completes the proof. ∎

As an immediate consequence of Theorem 1.4 we may order the Dirichlet eigenvalues lexicographically so that

$$\cdots \mu_{n-1} \preccurlyeq \mu_n \preccurlyeq \mu_{n+1} \preccurlyeq \cdots,$$

and μ_n is a simple eigenvalue if $|n| > N_\varphi$. Similarly, one can order the Neumann eigenvalues lexicographically

$$\cdots \nu_{n-1} \preccurlyeq \nu_n \preccurlyeq \nu_{n+1} \preccurlyeq \cdots,$$

and ν_n is simple if $|n| > N_\varphi$.

Lemma 4.5 *Suppose $\varphi \in FL_c^p$, $1 \leqslant p < \infty$, then for each $|n| > N_\varphi$ the functions*

$$U_\varphi \to \mathbb{C}, \quad \varphi \mapsto \mu_n, \qquad U_\varphi \to \mathbb{C}, \quad \varphi \mapsto \nu_n,$$

are analytic. ⋊

Proof. The projector P_n^{dir}, $|n| > N_\varphi$, is analytic on U_φ and its range $\mathcal{P}_n^{dir}$ is one dimensional and spanned by the eigenfunction corresponding to μ_n. Consequently,

$$\mu_n(\psi) = \operatorname{Tr}\Big(L(\psi^{dir})\,|_{\mathcal{P}_n^{dir}(\psi)}\Big), \qquad \psi \in U_\varphi,$$

and one argues as in the proof of Lemma 3.7 to obtain analyticity. The argument for ν_n is the same. ∎

5. Auxiliary D and N spectra

The auxiliary D and N spectra are defined as the spectra of the operator (1.1)

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}$$

acting as an unbounded operator on $L^2[0,1]$ with domains

$$\begin{aligned} \mathcal{D}_{d\star} &= \{f \in H_c^1[0,1] : (f_- + \mathrm{i}f_+)\,|_0 = 0 = (f_- + \mathrm{i}f_+)\,|_1\}, \\ \mathcal{D}_{n\star} &= \{g \in H_c^1[0,1] : (g_- - \mathrm{i}g_+)\,|_0 = 0 = (g_- - \mathrm{i}g_+)\,|_1\}. \end{aligned} \tag{1.10}$$

We follow a similar procedure as in the previous section to extend this definition to the case where φ is an element of the Fourier Lebesgue space FL_c^p. To this end we define for any $f \in FL_c^p(\mathbb{T}_1)$ the extensions

$$f^{d\star}(x) = \begin{cases} f(x), & 0 \leqslant x \leqslant 1, \\ \check{f}(2-x), & 1 \leqslant x \leqslant 2, \end{cases} \qquad \check{f} = (-\mathrm{i}f_+, \mathrm{i}f_-)$$

$$f^{n\star}(x) = \begin{cases} f(x), & 0 \leqslant x \leqslant 1, \\ -\check{f}(2-x), & 1 \leqslant x \leqslant 2, \end{cases}$$

and obtain the following result by a straightforward computation.

Lemma 5.1 *(i) Suppose $f \in \mathcal{D}_{d\star}$ solves $L(\varphi)f = \lambda f$ for some $\lambda \in \mathbb{C}$, then $f^{d\star}$ is an element of $H_c^1(\mathbb{T}_2)$ which solves $L(\varphi^{neu})f^{d\star} = \lambda f^{d\star}$.*

(ii) Suppose $f \in \mathcal{D}_{n\star}$ solves $L(\varphi)f = \lambda f$ for some $\lambda \in \mathbb{C}$, then $f^{n\star}$ is an element of $H_c^1(\mathbb{T}_2)$ which solves $L(\varphi^{neu})f^{n\star} = \lambda f^{n\star}$. ⋊

The Fourier coefficients of these extensions are given by

$$\begin{aligned} f^{d\star} &= \sum_{m\in\mathbb{Z}} \frac{1}{2}(\langle f_+, \mathrm{e}_m\rangle_{\mathbb{T}_1} + \mathrm{i}\langle f_-, \mathrm{e}_{-m}\rangle_{\mathbb{T}_1})e_m^+ + \frac{1}{2}(\langle f_-, \mathrm{e}_{-m}\rangle_{\mathbb{T}_1} - \mathrm{i}\langle f_+, \mathrm{e}_m\rangle_{\mathbb{T}_1})e_m^- \\ &= \sum_{m\in\mathbb{Z}} \frac{1}{2}\langle f, E_m^{d\star}\rangle_{\mathbb{T}_1} E_m^{dir}, \qquad E_m^{d\star} := \mathrm{i}e_m^+ + e_m^-, \end{aligned} \tag{1.11}$$

and

$$f^{n\star} = \sum_{m\in\mathbb{Z}} \frac{1}{2}\langle f, E_m^{n\star}\rangle_{\mathbb{T}_1} E_m^{n\star}, \qquad E_m^{n\star} \coloneqq \mathrm{i}e_m^+ - e_m^-. \tag{1.12}$$

Therefore, we introduce the spaces

$$FL_{d\star}^{s,p}(\mathbb{T}_2) = \Big\{ f = \sum_{m\in\mathbb{Z}} a_m E_m^{d\star} \;:\; (a_m)\in \ell^{s,p} \Big\},$$

$$FL_{n\star}^{s,p}(\mathbb{T}_2) = \Big\{ f = \sum_{m\in\mathbb{Z}} b_m E_m^{n\star} \;:\; (b_m)\in \ell^{s,p} \Big\},$$

which have similar properties as the spaces $FL_{dir}^{s,p}(\mathbb{T}_2)$ and $FL_{neu}^{s,p}(\mathbb{T}_2)$. Indeed, they are closed subspaces of $FL_c^{s,p}(\mathbb{T}_2)$ and $FL_c^{s,p}(\mathbb{T}_2) = FL_{d\star}^{s,p}(\mathbb{T}_2) \oplus FL_{d\star}^{s,p}(\mathbb{T}_2)$. Moreover, the mappings

$$\mathcal{D}_{d\star} \to FL_{d\star}^{1,2}(\mathbb{T}_2), \quad f\mapsto f^{d\star}, \qquad \mathcal{D}_{n\star} \to FL_{n\star}^{1,2}(\mathbb{T}_2), \quad f\mapsto f^{n\star},$$

are isomorphisms, and both extensions

$$FL_c^p(\mathbb{T}_1) \to FL_{d\star}^p(\mathbb{T}_2), \quad f\mapsto f^{d\star}, \qquad FL_c^p(\mathbb{T}_1) \to FL_{n\star}^p(\mathbb{T}_2), \quad f\mapsto f^{n\star},$$

are bounded maps for $1 < p < \infty$. Finally, we note that $L(\varphi^{neu})$ leaves them invariant – the proofs of these properties are completely analogous to the proofs found in Section 4.

We are now in a position to define the auxiliary spectra for the case where φ is in the Fourier Lebesgue space. The *auxiliary D spectrum of* $\varphi \in FL_c^p$ is defined as the spectrum of the operator $L(\varphi^{neu})|_{FL_{d\star}^p}$. The *auxiliary N spectrum of* $\varphi \in FL_c^p(\mathbb{T}_1)$ is defined as the spectrum of the operator $L(\varphi^{neu})|_{FL_{n\star}^p}$.

Proposition 5.2 *The auxiliary D and N spectra of any potential in* FL_c^p, $1<p<\infty$, *are discrete and there exists a neighborhood U of the potential in* FL_c^p *and an integer* $N>0$ *such that*

(i) for any $|n|>N$, *each* $\varphi\in U$ *has a simple auxiliary D resp. N eigenvalue in the disc*

$$D_n = \{\lambda\in\mathbb{C} : |\lambda - n\pi| < \pi/4\},$$

(ii) each $\varphi\in U$ *has exactly* $2N+1$ *auxiliary D resp. N eigenvalues in the box*

$$B_N = \{\lambda\in\mathbb{C} : |\Re\lambda| < N\pi + \pi/2, \quad |\Im\lambda| \leqslant (1+8\|\varphi\|_p)^p\},$$

(iii) there are no other auxiliary D or N eigenvalues.

(iv) If φ *is of real type, then all the eigenvalues are real valued.* ⋊

Proof. The proof is completely analogous the the proof of Theorem 1.4. Consider the operator $L(\varphi^{neu})$ acting on $FL_c^p(\mathbb{T}_2)$ with domain $FL_c^{1,p}(\mathbb{T}_2)$. By the analysis of Section 3 the spectrum of this operator is discrete and one can choose an neighborhood U_φ of φ and an integer $N_\varphi \geqslant 0$ so that the resolvent is analytic in λ on $\mathbb{C}\setminus(B_{N_\varphi}\cup\bigcup_{|n|>N_\varphi} D_n)$ for any $\psi\in U_\varphi$. Without loss one may assume that U_φ is connected and contains the origin. Moreover, the Cauchy-Riesz projectors $P_{N_\varphi}(\psi)$ and $P_n(\psi)$ for $|n|>N_\varphi$ defined in (1.4) are analytic on U_φ. In particular, the dimension of their range is constant on U_φ. Moreover, they leave the spaces $FL_{d\star}^p$ and $FL_{n\star}^p$ invariant. One easily checks that for the zero potential

$$\dim P_N(0)|_{FL_{d\star}^{1,p}} = 2N+1, \qquad \dim P_n(0)|_{FL_{d\star}^{1,p}} = 1, \quad |n|>N_\varphi,$$

$$\dim P_N(0)|_{FL_{n\star}^{1,p}} = 2N+1, \qquad \dim P_n(0)|_{FL_{n\star}^{1,p}} = 1, \quad |n|>N_\varphi,$$

which shows that D_n for $|n| > N_\varphi$ contains precisely one auxiliary D- and one auxiliary N-eigenvalue, while B_N contains precisely $2N+1$ auxiliary D- and $2N-1$ auxiliary N-eigenvalues when counted with multiplicities. ∎

6. Asymptotic behavior of the eigenvalues

We have seen in Section 3 that the periodic spectrum of $\varphi \in FL_c^p(\mathbb{T}_2)$ is discrete an consists of an bi-infinite sequence $(\lambda_n^\pm)_{n\in\mathbb{Z}}$ with asymptotic behavior $\lambda_n^\pm = n\pi + O(1)$ as $|n| \to \infty$. The goal for this section is to obtain a refined estimate of the asymptotic behavior reflecting the asymptotic behavior of the Fourier coefficients.

Proposition 6.1 *Locally uniformly on $FL_c^p(\mathbb{T}_2)$, $1 < p < \infty$,*

$$\lambda_n^\pm = n\pi + \ell_n^p.$$

More to the point, for any potential φ in $FL_c^p(\mathbb{T}_2)$, $1 < p < \infty$, there exists an integer $N_0 \geqslant 1$ so that for all $N \geqslant N_0$

$$\sum_{n\geqslant N} (|\lambda_n^+ - n\pi|^p + |\lambda_n^- - n\pi|^p) \leqslant c_p \left(\frac{\|\varphi\|_p^p}{N^{1\wedge(p-1)}} + \|R_{N/2}\varphi\|_p^p \right) (1 + \|\varphi\|_p^p) \|\varphi\|_p^p,$$

where $1 \wedge (p-1) \coloneqq \min(1, p-1)$, the constant c_p is absolute and depends only on p, and N_0 can be chosen locally uniformly in φ. ⋊

If φ is actually 1-periodic, then by the procedure laid out in Section 4, the Dirichlet and Neumann eigenvalues can be realized as the periodic eigenvalues of an appropriate 2-periodic extension of φ to $FL_c^p(\mathbb{T}_2)$. Proposition 6.1 applies to the latter yielding the following estimate of the asymptotic behavior of the Dirichlet and Neumann eigenvalues.

Corollary 6.2 *Locally uniformly on $FL_c^p(\mathbb{T}_2)$, $1 < p < \infty$,*

$$\mu_n = n\pi + \ell_n^p, \qquad \nu_n = n\pi + \ell_n^p,$$

as well as

$$\mu_n^\star = n\pi + \ell_n^p, \qquad \nu_n^\star = n\pi + \ell_n^p. \quad ⋊$$

Another immediate consequence of Proposition 6.1 is the following locally uniform estimate of the mid-points τ_n and the gap lengths γ_n of $\varphi \in FL_c^p$, $1 < p < \infty$,

$$\tau_n = n\pi + \ell_n^p, \qquad \gamma_n = \ell_n^p.$$

Since it does not create any additional effort, we prove that the asymptotic behavior of the gap lengths reflects the asymptotic behavior of the Fourier coefficients not only in FL_c^p but also for a larger family of spaces referred to *weighted Sobolev spaces* – see [28, 29] for an introduction. A *normalized, submultiplicative*, and *monotone weight* is a symmetric function $w\colon \mathbb{Z} \to \mathbb{R}$ with

$$w_n \geqslant 1, \qquad w_n = w_{-n}, \qquad w_{n+m} \leqslant w_n w_m, \qquad w_{|n|} \leqslant w_{|n|+1},$$

for all $n, m \in \mathbb{Z}$. The class of all such weights is denoted by $\mathcal{M}$ and $FL_c^{w,p}(\mathbb{T}_2)$ denotes the space of $FL_c^p(\mathbb{T}_2)$ functions φ with finite w-norm

$$\|\varphi\|_{w,p} := \left(\sum_{m\in\mathbb{Z}} w_m^p(|\varphi_m^-|^p + |\varphi_m^+|^p)\right)^{1/p}.$$

For example for any $s \geqslant 0$, the *Sobolev weight* $\langle n\pi\rangle^s$ gives rise to the usual Fourier Lebesgue space $FL_c^{s,p}(\mathbb{T}_2)$.

Proposition 6.3 *For any potential φ in $FL_c^{w,p}(\mathbb{T}_2)$ with $w \in \mathcal{M}$ and $1 < p < \infty$, there exists an integer $N_0 \geqslant 1$ so that for all $N \geqslant N_0$*

$$\sum_{n\geqslant N} w_{2n}^p|\gamma_n|^p \leqslant c_p\left(\frac{\|\varphi\|_{w,p}^p}{N^{1\wedge(p-1)}} + \|R_{N/2}\varphi\|_{w,p}^p\right)(1 + \|\varphi\|_{w,p}^p)\|\varphi\|_{w,p}^p$$

where c_p is an absolute constant depending only on p, and N_0 can be chosen locally uniformly in φ. ⋊

The proof of Propositions 6.1 and 6.3 is based on a *Lyapunov-Schmidt decomposition* introduced by Kappeler & Mityagin [28] – see also [24, 21]: For the free Zakharov-Shabat operator each $n\pi$, $n \in \mathbb{Z}$, is a double eigenvalue with eigenfunctions $e_n^+ = (0, \mathrm{e}^{\mathrm{i}n\pi x})$ and $e_n^- = (\mathrm{e}^{-\mathrm{i}n\pi x}, 0)$. Thus, for a nonzero potential $\varphi \in FL_c^p(\mathbb{T}_2)$, provided $|n|$ is sufficiently large, we expect exactly two eigenvalues which are close to $n\pi$ and whose eigenfunctions are close to the linear span of e_n^+ and e_n^-. This suggest to separate these modes from the others.

To this end, we introduce the complex strips

$$\mathfrak{U}_n := \{\lambda : |\Re\lambda - n\pi| \leqslant \pi/2\},$$

and consider the splitting

$$FL_c^{w,p}(\mathbb{T}_2) = \mathcal{P}_n \oplus \mathcal{Q}_n, \qquad \mathcal{P}_n := \mathrm{sp}\{e_n^+, e_n^-\}, \qquad \mathcal{Q}_n := \overline{\mathrm{sp}}\{e_k^+, e_k^- : k \neq n\}. \tag{1.13}$$

The projections onto $\mathcal{P}_n$ and $\mathcal{Q}_n$ are denoted by P_n and Q_n, respectively.

It turns out to be convenient to write the eigenvalue equation $L(\varphi)f = \lambda f$ in the form

$$A_\lambda f = \Phi f,$$

where $A_\lambda := \lambda - L(0)$ and $\Phi := \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}$. Clearly, A_λ is the Fourier multiplier $A_\lambda e_m^\pm = (\lambda - m\pi)e_m^\pm$, and hence leaves $\mathcal{Q}_n$ and $\mathcal{P}_n$ invariant. Therefore, by writing $f = u + v = P_n f + Q_n f$, we can decompose the equation $A_\lambda f = \Phi f$ into the two equations

$$A_\lambda u = P_n\Phi(u + v), \qquad A_\lambda v = Q_n\Phi(u + v),$$

called the P- and the Q-equation, respectively.

For $\lambda \in \mathfrak{U}_n$ and $m \neq n$ one has

$$\min_{\lambda\in\mathfrak{U}_n}|\lambda - m\pi| \geqslant |n - m| \geqslant 1,$$

hence the restriction of A_λ to $\mathcal{Q}_n$ is boundedly invertible with a uniform bound in $\lambda \in \mathfrak{U}_n$. We also note that, $R_0(\lambda) = A_\lambda^{-1}$ for $\lambda \notin \pi\mathbb{Z}$. Multiplying the Q-equation from the left by ΦA_λ^{-1} then gives

$$\Phi v = T_n\Phi(u + v),$$

with $T_n \coloneqq \Phi A_\lambda^{-1} Q_n$. The latter identity may be written as

$$(\mathrm{Id} - T_n)\Phi v = T_n \Phi u,$$

hence solving the Q equation amounts to inverting $(\mathrm{Id} - T_n)$.

To obtain the invertibility of $(\mathrm{Id} - T_n)$, we consider operator norms induced by *shifted weighted norms* – see e.g. [63]. For any $u \in FL^{w,p}(\mathbb{T}_2)$ with Fourier series $u = \sum_{m\in\mathbb{Z}} u_m \mathrm{e}_m$, the i-shifted $FL^{w,p}$-norm is given by

$$\|u\|_{w,p;i}^p \coloneqq \|u\mathrm{e}_i\|_{w,p}^p = \sum_{m\in\mathbb{Z}} w_{m+i}^p |u_m|^p.$$

Furthermore, for f an element of the space $FL_c^{w,p}(\mathbb{T}_2)$ with Fourier series

$$f = (f_-, f_+) = \sum_{n\in\mathbb{Z}} (f_n^- e_n^- + f_n^+ e_n^+) = \sum_{n\in\mathbb{Z}} (f_n^- \mathrm{e}_{-n}, f_n^+ \mathrm{e}_n),$$

the i-shifted norm is defined by

$$\|f\|_{w,p;i}^p \coloneqq \|f_-\|_{w,p;-i}^p + \|f_+\|_{w,p;i}^p = \sum_{m\in\mathbb{Z}} w_{m+i}^p (|f_m^-|^p + |f_m^+|^p).$$

Lemma 6.4 *If $\varphi \in FL_c^{w,p}(\mathbb{T}_2)$ with $w \in \mathcal{M}$ and $1 \leqslant p < \infty$, then for any $n \in \mathbb{Z}$ and any $\lambda \in \mathfrak{U}_n$,*

$$T_n = \Phi A_\lambda^{-1} Q_n \colon FL_c^{w,p}(\mathbb{T}_2) \to FL_c^{w,p}(\mathbb{T}_2)$$

is bounded and satisfies for any $i \in \mathbb{Z}$ the estimate

$$\|T_n f\|_{w,p;i} \leqslant c_p \|\varphi\|_{w,p} \|f\|_{w,p;-i},$$

where c_p is an absolute constant depending only on p. For $p = 2$ we can choose $c_p = 2$. ⋊

Proof. Write $T_n f = \Phi g$ with $g = A_\lambda^{-1} Q_n f$. Since the restriction of A_λ to $\mathcal{Q}_n$ is boundedly invertible, the function

$$g = A_\lambda^{-1} Q_n f = \sum_{m\neq n} \left(\frac{f_m^-}{\lambda - m\pi} e_m^- + \frac{f_m^+}{\lambda - m\pi} e_m^+ \right) = (g_-, g_+)$$

is well defined. By Hölder's inequality and Lemma B.1, we obtain for the weighted ℓ^1-norm with p' conjugated to p, $p > 1$,

$$\|g_+ \mathrm{e}_{-i}\|_{\ell_w^1} = \sum_{m\neq n} \frac{w_{m-i}|f_m^+|}{|\lambda - m\pi|} \leqslant \left(\sum_{m\neq n} \frac{1}{|n-m|^{p'}} \right)^{1/p'} \|f_+\|_{w,p;-i} \leqslant c_p \|f_+\|_{w,p;-i},$$

uniformly for $\lambda \in \mathfrak{U}_n$ – with obvious modifications for the case $p = 1$. Similarly, one shows that $\|g_- \mathrm{e}_i\|_{\ell_w^1} \leqslant c_p \|f_-\|_{w,p;i}$. Since

$$\|T_n f\|_{w,p;i}^p = \|\Phi g\|_{w,p;i}^p = \|\varphi_- g_+ \mathrm{e}_{-i}\|_{w,p}^p + \|\varphi_+ g_- \mathrm{e}_i\|_{w,p}^p,$$

we can apply Young's inequality (A.3) to estimate the product of φ_- with $g_+\mathrm{e}_{-i}$, giving

$$\|T_n f\|_{w,p;i}^p \leqslant \|\varphi\|_{w,p}^p \left(\|g_+ \mathrm{e}_{-i}\|_{\ell_w^1}^p + \|g_- \mathrm{e}_i\|_{\ell_w^1}^p \right) \leqslant c_p^p \|\varphi\|_{w,p}^p \|f\|_{w,p;-i}^p. \quad \blacksquare$$

Note that $T_n f$ is estimated with a shifted $FL_c^{w,p}$-norm where the sign of the shift is opposite to the sign of the shifted $FL_c^{w,p}$-norm of f. This fact will be crucial in the following. In particular, T_n^2 is bounded with respect to the shifted norm and it turns out that $\|T_n^2\|_{w,p;n} = o(1)$ as $|n| \to \infty$. Using a Neumann series, we then obtain the bounded invertibility of $(\mathrm{Id} - T_n)$ for $|n|$ sufficiently large, which solves the Q-equation.

Lemma 6.5 *If $\varphi \in FL_c^{w,p}(\mathbb{T}_2)$ with $w \in \mathcal{M}$ and $1 \leqslant p < \infty$, then for any $n \in \mathbb{Z}$ and any $\lambda \in \mathfrak{U}_n$*

$$\|T_n^2\|_{w,p;n} \leqslant c_p \|\varphi\|_{w,p} \left(\frac{\|\varphi\|_{w,p}}{\langle n\rangle^{1/p}} + \frac{\|R_n\varphi\|_{w,p}}{w_n} \right),$$

where $R_n\varphi$ denotes the nth remainder of φ

$$R_n\varphi \coloneqq \sum_{|k|\geqslant|n|} (\varphi_k^- e_k^- + \varphi_k^+ e_k^+). \qquad \rtimes$$

Proof. As in the preceding lemma, write $T_n^2 f = \Phi g$ with

$$g \coloneqq A_\lambda^{-1} Q_n \Phi A_\lambda^{-1} Q_n f.$$

A straightforward computation yields

$$g = \sum_{k,l\neq n} \left(\frac{\varphi_{k+l}^-}{\lambda - k\pi} \frac{f_l^+}{\lambda - l\pi} e_k^- + \frac{\varphi_{k+l}^+}{\lambda - k\pi} \frac{f_l^-}{\lambda - l\pi} e_k^+ \right) = (g_-, g_+),$$

and our aim is to estimate the weighted ℓ^1-norm

$$\|g_+ \mathrm{e}_{-n}\|_{1,w} \leqslant \sum_{k,l\neq n} w_{k-n} \frac{|\varphi_{k+l}^+|}{|n-k|} \frac{|f_l^-|}{|n-l|}.$$

To proceed, we split up the latter sum into two parts $\Sigma_\flat + \Sigma_\natural$ defined by the index sets

$$I_\flat \coloneqq \{k, l \neq n : |n-k| > |n|/2 \quad \text{or} \quad |n-l| > |n|/2\},$$
$$I_\natural \coloneqq \{k, l \neq n : |n-l| \leqslant |n|/2 \quad \text{and} \quad |n-k| \leqslant |n|/2\} = I_\flat^c.$$

Using Hölder's inequality together with the submultiplicity and the symmetry of the weight w, we obtain for the first term

$$\Sigma_\flat = \sum_{I_\flat} w_{k-n} \frac{|\varphi_{k+l}^+|}{|n-k|} \frac{|f_l^-|}{|n-l|} \leqslant \left\| \left(\frac{1}{|n-k|} \frac{1}{|n-l|} \right)_{(k,l)\in I_\flat} \right\|_{\ell^{p'}\times\ell^{p'}} \left(\sum_{k,l} w_{k+l}^p |\varphi_{k+l}^+|^p \, w_{-l-n}^p |f_l^-|^p \right)^{\frac{1}{p}}$$
$$\leqslant \frac{c_p^2}{\langle n/2\rangle^{1/p}} \|\varphi_+\|_{w,p} \|f_-\|_{w,p;-n},$$

where we used Lemma 2.25 and Young's inequality (B.1) in the second step.

Conversely, for (k,l) taken from $I_\natural$ we have $1 \leqslant |n-k|, |n-l| \leqslant |n|/2$ and hence

$$|k+l| \geqslant 2|n| - |n-k| - |n-l| \geqslant |n| \geqslant |n-k|, \qquad |n+l| \geqslant |n|.$$

By the monotonicity of the weight $w_{k-n}w_n \leqslant w_{k+l}w_{-l-n}$, thus we obtain with Hölder's and Young's inequality

$$\Sigma_\natural = \sum_{I_\natural} w_{k-n} \frac{|\varphi_{k+l}^+|}{|n-k|} \frac{|f_l^-|}{|n-l|}$$
$$\leqslant \frac{1}{w_n} \left\| \left(\frac{1}{|n-k|} \frac{1}{|n-l|} \right)_{(k,l)\in I_\natural} \right\|_{\ell^{p'}\times\ell^{p'}} \left(\sum_{I_\natural} w_{k+l}^p |\varphi_{k+l}^+|^p \, w_{-l-n}^p |f_l^-|^p \right)^{\frac{1}{p}}$$
$$\leqslant \frac{c_p^2}{w_n} \|R_n\varphi_+\|_{w,p} \|f_-\|_{w,p;-n}.$$

Both estimates together yield

$$\|g_+ \mathrm{e}_{-n}\|_{1,w} \leqslant 2^{1/p} c_p^2 \left(\frac{\|\varphi_+\|_{w,p}}{\langle n\rangle^{1/p}} + \frac{\|R_n\varphi_+\|_{w,p}}{w_n} \right) \|f_-\|_{w,p;-n},$$

and, in a similar fashion,

$$\|g_-\mathrm{e}_n\|_{1,w} \leqslant 2^{1/p} c_p^2 \left(\frac{\|\varphi_-\|_{w,p}}{\langle n\rangle^{1/p}} + \frac{\|R_n\varphi_-\|_{w,p}}{w_n} \right) \|f_+\|_{w,p;n}.$$

Since $\|T_n^2 f\|_{w,p;n} = \|\Phi g\|_{w,p;n}$, we obtain by the Young inequality (Lemma A.7)

$$\begin{aligned} \|T_n^2 f\|_{w,p;n}^p &\leqslant \|\varphi\|_{w,p}^p \left(\|g_+\mathrm{e}_{-n}\|_{1,w}^p + \|g_-\mathrm{e}_n\|_{1,w}^p\right) \\ &= 2c_p^{2p}\|\varphi\|_{w,p}^p \left(\frac{\|\varphi\|_{w,p}}{\langle n\rangle^{1/p}} + \frac{\|R_n\varphi\|_{w,p}}{w_n} \right)^p \|f\|_{w,p;n}^p. \end{aligned}$$ ∎

Consequently, T_n^2 is a contraction on $FL_c^{w,p}(\mathbb{T}_2)$ for all $|n|$ sufficiently large. The threshold of $|n|$ can be chosen locally uniformly in φ on $FL_c^{w,p}$ due to the $R_n\varphi$ term. We choose a neighborhood U_φ in $FL_c^{w,p}$ and an integer $N_{w,\varphi,p} \geqslant 1$ such that for $|n| \geqslant N_{w,\varphi,p}$ and on all of U_φ

$$\|T_n^2\|_p,\ \|T_n^2\|_{w,p;n} \leqslant \frac{1}{2}.$$

To simplify notation, write $N_{\varphi,p}$ in the case $w_n \equiv 1$. In view of

$$\hat{T}_n := (\mathrm{Id} - T_n)^{-1} = (\mathrm{Id} + T_n)(\mathrm{Id} - T_n^2)^{-1},$$

one then finds for every $|n| \geqslant N_{w,\varphi,p}$ a unique solution

$$\Phi v = \hat{T}_n T_n \Phi u$$

of the Q-equation. In turn, as $I + \hat{T}_n T_n = \hat{T}_n$, the P-equation yields

$$A_\lambda u = P_n(\mathrm{Id} + \hat{T}_n T_n)\Phi u = P_n \hat{T}_n \Phi u.$$

Writing the latter as

$$S_n u = 0, \qquad S_n \colon \mathcal{P}_n \to \mathcal{P}_n, \quad u \mapsto (A_\lambda - P_n\hat{T}_n\Phi)u,$$

we immediately conclude that there exists the following relationship.

Lemma 6.6 *For $\varphi \in FL_c^p(\mathbb{T}_2)$ with $1 \leqslant p < \infty$ and $|n| \geqslant N_{\varphi,p}$, a complex number $\lambda \in \mathfrak{U}_n$ is a periodic eigenvalue of φ if and only if the determinant of S_n vanishes.* ⋊

Proof. Suppose $f \in FL_c^{1,p}$ and $Lf = \lambda f$, then by the preceding discussion $S_n u = 0$. Conversely, define for $u \in \mathcal{P}_n$,

$$v = A_\lambda^{-1} Q_n \hat{T}_n \Phi u,$$

then v is an element of $FL_c^{1,p} \cap \mathcal{Q}_n$ since A_λ^{-1} is 1-smoothing and leaves $\mathcal{Q}_n$ invariant. In particular, $\Phi v = T_n \hat{T}_n \Phi u$ is well defined, and it follows with $\hat{T}_n = I + T_n\hat{T}_n$ that

$$A_\lambda v = Q_n \hat{T}_n \Phi u = Q_n \Phi u + Q_n T_n \hat{T}_n \Phi u = Q_n \Phi(u+v),$$

so the Q-equation is automatically satisfied. Moreover, if $S_n u = 0$, then

$$A_\lambda u = P_n \hat{T}_n \Phi u = P_n(I + T_n\hat{T}_n)\Phi u = P_n\Phi(u+v).$$

Hence also the P-equation is satisfied, and λ is an eigenvalue of L with eigenfunction $f = u + v$. ∎

Recall that P_n is the projection onto the two-dimensional space $\mathcal{P}_n$. The matrix representation of an operator B on $\mathcal{P}_n$ is given by

$$(\langle Be_n^{\pm}, e_n^{\pm}\rangle)_{\pm,\pm}.$$

Therefore, we find for S_n the representation

$$A_\lambda = \begin{pmatrix} \lambda - n\pi & \\ & \lambda - n\pi \end{pmatrix}, \qquad P_n \hat{T}_n \Phi = \begin{pmatrix} a_n^+ & b_n^+ \\ b_n^- & a_n^- \end{pmatrix},$$

with the coefficients of the latter matrix given by

$$a_n^+ \coloneqq \langle \hat{T}_n \Phi e_n^+, e_n^+\rangle, \qquad\qquad b_n^+ \coloneqq \langle \hat{T}_n \Phi e_n^-, e_n^+\rangle,$$
$$b_n^- \coloneqq \langle \hat{T}_n \Phi e_n^+, e_n^-\rangle, \qquad\qquad a_n^- \coloneqq \langle \hat{T}_n \Phi e_n^-, e_n^-\rangle.$$

We point out that these coefficients depend on λ and φ. It has been observed in [37, 18] that these coefficients reflect certain symmetries of the Fourier coefficients of the potential φ.

Lemma 6.7 *Suppose $\varphi \in FL_c^p(\mathbb{T}_2)$, $1 \leqslant p < \infty$, and $|n| \geqslant N_{\varphi,p}$. Then for any $\lambda \in \mathfrak{U}_n$,*

(i) $a_n^+ = a_n^- \equiv a_n$,

(ii) $a_n(\bar{\lambda}) = \overline{a_n(\lambda)}$ *and, if* $\varphi^* = \pm\varphi$, *then* $b_n^+(\bar{\lambda}) = \pm\overline{b_n^-(\lambda)}$. ⋊

Proof. (a): Recall that $T_n = \Phi A_\lambda^{-1} Q_n$, and denote the complex conjugation of operators by $\overline{B}u \coloneqq \overline{B\bar{u}}$. From evaluating the bounded diagonal operators A_λ^{-1} and Q_n at $e_m^{\pm}$, and using the identity $\overline{e_m^{\pm}} = Pe_m^{\mp}$, we conclude

$$(A_\lambda^{-1})^* = P\overline{A_\lambda^{-1}}P = A_{\bar{\lambda}}^{-1}, \quad Q_n^* = P\overline{Q_n}P = Q_n, \qquad P \coloneqq \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Since A_λ^{-1} leaves $\mathcal{Q}_n$ invariant, and $P^2 = I$, we find $(A_\lambda^{-1}Q_n)^* = P\overline{A_\lambda^{-1}Q_n}P$. Together with $\Phi^* = P\overline{\Phi}P$ this gives

$$(T_n\Phi)^* = \Phi^*(A_\lambda^{-1}Q_n)^*\Phi^* = P\overline{T_n\Phi}P.$$

Inspecting the Neumann expansion of $\hat{T}_n\Phi$ yields $(\hat{T}_n\Phi)^* = P\overline{\hat{T}_n\Phi}P$, thus

$$\begin{aligned} a_n^+ = \langle \hat{T}_n\Phi e_n^+, e_n^+\rangle &= \langle e_n^+, (\hat{T}_n\Phi)^* e_n^+\rangle \\ &= \langle Pe_n^+, \overline{\hat{T}_n\Phi} Pe_n^+\rangle = \langle \hat{T}_n\Phi e_n^-, e_n^-\rangle = a_n^-. \end{aligned}$$

b): Suppose $\varphi^* = \pm\varphi$. Then we have $\Phi^* = \pm\Phi$,

$$\overline{T_n(\lambda)} = \overline{\Phi A_\lambda^{-1} Q_n} = \pm P\Phi A_{\bar{\lambda}}^{-1} Q_n P = \pm P T_n(\bar{\lambda}) P,$$

and using the identity $\hat{T}_n = (\mathrm{Id} + T_n)(\mathrm{Id} - T_n^2)^{-1}$ we find

$$(\hat{T}_n(\lambda)\Phi)^* = \pm P\overline{(\mathrm{Id} + T_n(\lambda))(\mathrm{Id} - T_n(\lambda)^2)^{-1}}P\Phi = \pm(\mathrm{Id} \pm T_n(\bar{\lambda}))(\mathrm{Id} - T_n(\bar{\lambda})^2)^{-1}\Phi.$$

Since $\Phi = \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}$ is anti-diagonal while A_λ and Q_n are diagonal, we conclude that T_n is anti-diagonal. Hence T_n^2 is diagonal and we obtain

$$\begin{aligned} a_n^+(\bar{\lambda}) &= \langle \hat{T}_n(\bar{\lambda})\Phi e_n^+, e_n^+\rangle \\ &= \pm\langle e_n^+, (\mathrm{Id} \pm T_n(\lambda))(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^+\rangle \\ &= \langle e_n^+, T_n(\lambda)(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^+\rangle \\ &= \langle e_n^+, \hat{T}_n(\lambda)\Phi e_n^+\rangle = \overline{a_n^+(\lambda)}. \end{aligned}$$

For the case of b_n^+ we get on the other hand,

$$
\begin{aligned}
b_n^+(\bar\lambda) = &\ \langle \hat T_n(\bar\lambda)\Phi e_n^-, e_n^+\rangle \\
&= \pm\langle e_n^-, (\mathrm{Id} \pm T_n(\lambda))(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^+\rangle \\
&= \pm\langle e_n^-, (\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^+\rangle \\
&= \pm\overline{b_n^-(\lambda)}. \qquad \blacksquare
\end{aligned}
$$

It follows that S_n may be written as

$$
S_n(\lambda) = \begin{pmatrix} \lambda - n\pi - a_n & -b_n^+ \\ -b_n^- & \lambda - n\pi - a_n \end{pmatrix}.
$$

Since T_n and Φ are anti-diagonal while Id and T_n^2 are diagonal, all even terms $\langle T_n^{2k}\Phi e_n^+, e_n^+\rangle$ in the expansion of a_n vanish. Using $\hat T_n = (\mathrm{Id} + T_n)(\mathrm{Id} - T_n^2)^{-1}$, we thus conclude

$$
a_n = \langle \hat T_n \Phi e_n^+, e_n^+\rangle = \langle T_n(\mathrm{Id} - T_n^2)^{-1}\Phi e_n^+, e_n^+\rangle. \tag{1.14}
$$

On the other hand, all odd terms in the expansion of b_n vanish, such that

$$
b_n^\pm - \varphi_{2n}^\pm = \langle (\hat T_n - I)\Phi e_n^\mp, e_n^\pm\rangle = \langle T_n^2(\mathrm{Id} - T_n^2)^{-1}\Phi e_n^\mp, e_n^\pm\rangle. \tag{1.15}
$$

We introduce the following notion for the sup-norm of a complex valued function on a domain $U \subset \mathbb{C}$,

$$
|f|_U := \sup_{\lambda\in U}|f(\lambda)|.
$$

Lemma 6.8 *Suppose $\varphi \in FL_c^{w,p}(\mathbb{T}_n)$ with $w \in \mathcal{M}$ and $1 < p < \infty$. Then for any $|n| \geqslant N_{\varphi,w,p}$ the coefficients a_n and $b_n^\pm$ are analytic functions on U_n with bounds*

(i)
$$
\sum_{|n|\geqslant N} |a_n|_{\mathfrak{U}_n}^p \leqslant c_p\left(\frac{\|\varphi\|_p^p}{N^{1\wedge(p-1)}} + \|R_{N/2}\varphi\|_p^p\right)\|\varphi\|_p^p.
$$

(ii)
$$
\sum_{|n|\geqslant N} w_{2n}|b_n^\pm - \varphi_{2n}^\pm|_{\mathfrak{U}_n} \leqslant c_p\left(\frac{\|\varphi_+\|_{w,p}^{2p}}{N^{1\wedge(p-1)}} + \|R_{N/2}\varphi\|_{w,p}^{2p}\right)\|\varphi_\pm\|_{w,p}^p.
$$

Here $1\wedge(p-1) := \min(1, p-1)$. $\rtimes$

Proof. Each term in the series expansions (1.14) of a_n and (1.15) of $b_n^\pm$ is an analytic function in λ on $\mathfrak{U}_n$. Since $\|T_n^2\|_{w,p;n} \leqslant 1/2$, the series expansions converge uniformly on $\mathfrak{U}_n$, hence a_n and $b_n^\pm$ are analytic functions on $\mathfrak{U}_n$. To obtain the estimates (i) and (ii), we introduce the sequence $u_n = (\mathrm{Id} - T_n^2)^{-1}\Phi e_n^+$. This sequence uniformly bounded in n, namely

$$
\|u_n\|_{w,p;n} \leqslant \|(\mathrm{Id} - T_n^2)^{-1}\|_{w,p;n}\|\Phi e_n^+\|_{w,p;n} \leqslant 2\|\varphi_- \mathrm{e}_n\|_{w,p;-n} = 2\|\varphi_-\|_{w,p}.
$$

The Fourier series expansion of u_n is denoted by $\sum_{m\in\mathbb{Z}}(u_{m;n}^- e_m^- + u_{m;n}^+ e_m^+)$.

(i) A straightforward computation shows that

$$
a_n = \langle T_n u_n, e_n^+\rangle = \sum_{m\neq n}\frac{\varphi_{n+m}^+}{\lambda - m\pi}u_{m;n}^-.
$$

To estimate the ℓ^p-norm of the sequence (a_n), we use Hölder's inequality together with the estimate $\|u_n\|_p \leqslant \|u_n\|_{w,p;n} \leqslant 2\|\varphi_-\|_{w,p}$,

$$\begin{aligned}\sum_{|n|\geqslant N} |a_n|^p_{\mathfrak{U}_n} &\leqslant \sum_{|n|\geqslant N}\left(\sum_{m\neq n}\frac{|\varphi^+_{n+m}|^{p'}}{|m-n|^{p'}}\right)^{p/p'}\left(\sum_{m\in\mathbb{Z}}|u^-_{m;n}|^p\right)^{p/p}\\ &\leqslant 2^p\|\varphi_-\|_p^p\sum_{|n|\geqslant N}\left(\sum_{m\neq n}\frac{|\varphi^+_{n+m}|^{p'}}{|m-n|^{p'}}\right)^{p/p'}.\end{aligned}$$

To proceed, we split up the inner sum and treat the cases where $|m-n| > |n|/2$ and the remainder separately. If $p \geqslant 2$, then $p/p' \geqslant 1$, hence we can apply Young's inequality (B.1) and subsequently Lemma B.1 to obtain

$$\sum_{|n|\geqslant N}\left(\sum_{|m-n|\geqslant|n|/2}\frac{|\varphi^+_{n+m}|^{p'}}{|m-n|^{p'}}\right)^{p/p'} \leqslant \left(\sum_{|l|\geqslant N/2}\frac{1}{|l|^{p'}}\right)^{p/p'}\left(\sum_{m\in\mathbb{Z}}|\varphi^+_m|^p\right) \leqslant c_p\frac{\|\varphi_+\|_p^p}{N}.$$

On the other hand, $p/p' < 1$ for $1 < p < 2$ and since $(|a|+|b|)^{p/p'} \leqslant |a|^{p/p'} + |b|^{p/p'}$ for all $a, b \in \mathbb{C}$, we get

$$\sum_{|n|\geqslant N}\left(\sum_{|m-n|\geqslant|n|/2}\frac{|\varphi^+_{n+m}|^{p'}}{|m-n|^{p'}}\right)^{p/p'} \leqslant \sum_{|n|\geqslant N}\left(\sum_{|m-n|\geqslant|n|/2}\frac{|\varphi^+_{n+m}|^{p}}{|m-n|^{p}}\right) \leqslant c_p\frac{\|\varphi_+\|_p^p}{N^{p-1}}.$$

To estimate the remainder in the case $p \geqslant 2$, that is $p/p' \geqslant 1$, we use Young's inequality together with the fact that $0 < |m-n| < |n|/2$ implies $|m+n| \geqslant |n|$ to obtain

$$\sum_{|n|\geqslant N}\left(\sum_{\substack{|m-n|<|n|/2\\ m\neq n}}\frac{|\varphi^+_{n+m}|^{p'}}{|m-n|^{p'}}\right)^{p/p'} \leqslant \left(\sum_{|l|\neq 0}\frac{1}{|l|^{p'}}\right)^{p/p'}\left(\sum_{|m|\geqslant N}|\varphi^+_m|^p\right) \leqslant c_p\|R_N\varphi_+\|_p^p.$$

If $1 < p \leqslant 2$, then we use the simple inequality to get

$$\sum_{|n|\geqslant N}\left(\sum_{\substack{|m-n|<|n|/2\\ m\neq n}}\frac{|\varphi^+_{n+m}|^{p'}}{|m-n|^{p'}}\right)^{p/p'} \leqslant \sum_{|n|\geqslant N}\left(\sum_{\substack{|m-n|<|n|/2\\ m\neq n}}\frac{|\varphi^+_{n+m}|^{p}}{|m-n|^{p}}\right) \leqslant c_p\|R_N\varphi_+\|_p^p.$$

This proves the estimate of the coefficients a_n.

(ii) Since the case of estimating b_n^+ and b_n^- is similar, we concentrate on b_n^-. We first note that

$$b_n^- - \varphi^-_{2n} = \langle T_n^2 u_n, e_n^-\rangle = \sum_{k,l\neq n} w_{2n}\frac{\varphi^-_{n+l}}{\lambda - l\pi}\frac{\varphi^+_{l+k}}{\lambda - k\pi}u^-_{k;n}.$$

By the submultiplicity and the symmetry of the weight we thus find

$$w_{2n}|\langle T_n^2 u_n, e_n^-\rangle| \leqslant \sum_{k,l\neq n}\frac{w_{n+l}|\varphi^-_{n+l}|}{|n-l|}\frac{w_{l+k}|\varphi^+_{l+k}|}{|n-k|}w_{k+n}|u^-_{k;n}|.$$

The latter sum is split up into three parts defined by the index sets

$$\begin{aligned}I_k^n &\coloneqq \{k, l \neq n : |n-k| > |n|/2\},\\ I_l^n &\coloneqq \{k, l \neq n : |n-l| > |n|/2\},\\ I_\natural^n &\coloneqq \{k, l \neq n : 1 \leqslant |n-k|, |n-l| \leqslant |n|/2\}.\end{aligned}$$

Let I^n denote any of these index sets, then by Hölder's inequality with $1/p + 1/p' = 1$

$$\sum_{I^n}\frac{w_{n+l}|\varphi^-_{n+l}|}{|n-l|}\frac{w_{l+k}|\varphi^+_{l+k}|}{|n-k|}w_{k+n}|u^-_{k;n}| \leqslant \Sigma_{I^n}\|\varphi_-\|_{w,p}\|u_n\|_{w,p;n}, \tag{1.16}$$

where

$$\Sigma_{I^n} = \left(\sum_{I^n} \frac{w_{l+k}^{p'} |\varphi_{l+k}^+|^{p'}}{|n-l|^{p'} |n-k|^{p'}} \right)^{1/p'}.$$

Note that if $I^n = I^n_\sharp$, then the second factor in (1.16) is actually $\|R_{n/2}\varphi_-\|_{w,p}$ instead of $\|\varphi_-\|_{w,p}$.

We first consider the case where $p \geqslant 2$. For I^n_k by Young's inequality B.3 with parameters $\gamma, p_2 = p$ and $\beta, \alpha, p_2, p_3 = p'$ we obtain

$$\sum_{|n|\geqslant N} |\Sigma_{I^n_k}|^p \leqslant \left(\sum_{|k|>N/2} \frac{1}{|k|^{p'}} \right)^{p/p'} \left(\sum_{l\neq 0} \frac{1}{|l|^{p'}} \right)^{p/p'} \left(\sum_{m\in\mathbb{Z}} w_m^p |\varphi_m^+|^p \right) \leqslant \frac{c_p}{N} \|\varphi_+\|_{w,p}^p.$$

By analogous arguments, one obtains for I^n_l

$$\sum_{|n|\geqslant N} |\Sigma_{I^n_l}|^p \leqslant \left(\sum_{k\neq 0} \frac{1}{|k|^{p'}} \right)^{p/p'} \left(\sum_{|l|>N/2} \frac{1}{|l|^{p'}} \right)^{p/p'} \left(\sum_{m\in\mathbb{Z}} w_m^p |\varphi_m^+|^p \right) \leqslant \frac{c_p}{N} \|\varphi_+\|_{w,p}^p.$$

Finally, note that $|l+k| \geqslant 2|n| - |n-k| - |n-l| \geqslant |n|$ for $(k,l) \in I^n_\sharp$ and hence

$$\sum_{|n|\geqslant N} |\Sigma_{I^n_\natural}|^p \leqslant \left(\sum_{k\neq 0} \frac{1}{|k|^{p'}} \right)^{p/p'} \left(\sum_{l\neq 0} \frac{1}{|l|^{p'}} \right)^{p/p'} \left(\sum_{|m|\geqslant N} w_m^p |\varphi_m^+|^p \right) \leqslant c_p \|R_N\varphi_+\|_{w,p}^p.$$

Altogether we thus have if $p \geqslant 2$

$$\sum_{|n|\geqslant N} w_{2n}^p |b_n^- - \varphi_{2n}^-|^p \leqslant c_p \left(\frac{\|\varphi\|_{w,p}^{2p}}{N} + \|R_N\varphi\|_{w,p}^{2p} \right) \|\varphi_-\|_{w,p}^p.$$

On the other hand, if $1 < p < 2$, then the simple inequality $(|a|+|b|)^{p/p'} \leqslant |a|^{p/p'} + |b|^{p/p'}$ gives

$$\sum_{|n|\geqslant N} \Sigma_{I^n}^p = \sum_{|n|\geqslant N} \left(\sum_{I^n} \frac{w_{l+k}^{p'} |\varphi_{l+k}^+|^{p'}}{|n-l|^{p'} |n-k|^{p'}} \right)^{p/p'} \leqslant \sum_{|n|\geqslant N} \sum_{I^n} \frac{w_{l+k}^p |\varphi_{l+k}^+|^p}{|n-l|^p |n-k|^p},$$

and hence

$$\sum_{|n|\geqslant N} |\Sigma_{I^n_k}|^p + \sum_{|n|\geqslant N} |\Sigma_{I^n_l}|^p \leqslant \frac{c_p}{N^{p-1}} \|\varphi_+\|_{w,p}^p, \qquad \sum_{|n|\geqslant N} |\Sigma_{I^n_\natural}|^p \leqslant c_p \|R_N\varphi_+\|_{w,p}^p.$$

This proves the claimed estimate for b_n^-. ∎

In consequence, the determinant of S_n

$$\det S_n = (\lambda - n\pi - a_n)^2 - b_n^+ b_n^-$$

is analytic on $\mathfrak{U}_n$ and close to $(\lambda - n\pi)^2$ provided $|n|$ is sufficiently large, so Rouche's Theorem can be used to estimate its roots.

Lemma 6.9 *Let $\varphi \in F L_c^p(\mathbb{T}_2)$, $1 < p < \infty$, then there exists $N_\star \geqslant N_{\varphi,p}$ so that for any $|n| \geqslant N_\star$, the determinant of S_n has exactly two complex roots ξ_n^+, ξ_n^- in $\mathfrak{U}_n$, which are contained in the disc*

$$D_n := \left\{ |\lambda - n\pi| < \frac{\pi}{4} \right\},$$

and satisfy $|\xi_n^+ - \xi_n^-|^2 \leqslant 6|b_n^+ b_n^-|_{\mathfrak{U}_n}$ as well as for any $N \geqslant N_\star$

$$\sum_{n\geqslant N} (|\xi_n^+ - n\pi|^p + |\xi_n^+ - n\pi|^p) \leqslant c_p \left(\frac{\|\varphi\|_p^p}{N^{1\wedge(p-1)}} + \|R_{N/2}\varphi\|_p^p \right) (1 + \|\varphi\|_p^p) \|\varphi\|_p^p.$$

Here $N_\star$ can be chosen locally uniformly in φ. ⋊

Proof. By the the estimates of the preceding lemma we can choose $N_\star \geqslant N_{\varphi,p}$ locally uniformly in φ so that

$$|a_n|_{\mathfrak{U}_n} \leqslant \pi/32, \qquad |b_n^\pm|_{\mathfrak{U}_n} \leqslant \pi/16,$$

where we used that $|b_n^\pm| \leqslant \|R_{2n}\varphi\|_p + |b_n^\pm - \varphi_{2n}^\pm|$. Consequently,

$$|a_n|_{\mathfrak{U}_n} + |b_n^+ b_n^-|_{\mathfrak{U}_n}^{1/2} < \inf_{\lambda \in \mathfrak{U}_n \setminus D_n} |\lambda - n\pi| = \frac{\pi}{4}.$$

It follows from Rouche's Theorem that the function $h = \lambda - n\pi - a_n$ has a single root in D_n, just as $\lambda - n\pi$. Furthermore, h^2 and $\det S_n$ have the same number of roots in D_n, namely two when counted with multiplicity, while $\det S_n$ clearly has no root in $\mathfrak{U}_n \setminus D_n$.

To estimate the roots, we write $\det S_n = g_+ g_-$ with

$$g_\pm = \lambda - \pi n - a_n \mp \sigma_n, \qquad \sigma_n = \sqrt{b_n^+ b_n^-},$$

where the branch of the root is immaterial. Each root ξ_n of $\det S_n$ is either a root of g_+ or g_-, respectively, and thus satisfies $\xi_n = \pi n + a_n(\xi_n) \pm \sigma_n(\xi_n)$, hence the estimate of $\sum_{n \geqslant N} |\xi_n - n\pi|^p$ follows from the preceding Lemma. Moreover,

$$\begin{aligned} |\xi_n^+ - \xi_n^+| &\leqslant |a_n(\xi_n^+) - a_n(\xi_n^-)| + |\sigma_n(\xi_+) \pm \sigma_n(\xi_-)| \\ &\leqslant |\partial_\lambda a_n|_{D_n} |\xi_n^+ - \xi_n^-| + 2|\sigma_n|_{\mathfrak{U}_n}. \end{aligned}$$

Since $\operatorname{dist}(D_n, \partial\mathfrak{U}_n) \geqslant \pi/2 - \pi/4$, we obtain from Cauchy's estimate

$$|\partial_\lambda a_n|_{D_n} \leqslant \frac{|a_n|_{\mathfrak{U}_n}}{\operatorname{dist}(D_n, \partial\mathfrak{U}_n)} \leqslant \frac{\pi/32}{\pi/2 - \pi/4} \leqslant \frac{1}{8},$$

which implies $|\xi_n^+ - \xi_n^-|^2 \leqslant 6|b_n^+ b_n^-|_{\mathfrak{U}_n}$. ∎

We are now in a position to prove Propositions 6.1 and 6.3 which imply Theorem 1.2 and 1.5.

Proof of Propositions 6.1 & 6.3. Given $\varphi \in FL_c^p(\mathbb{T}_2)$, we can choose according to Lemma 6.9 an integer $N_\star \geqslant N_{\varphi,p}$ locally uniformly in φ so that for every $|n| \geqslant N_\star$ the function $\det S_n$ has precisely two roots ξ_n^+ and ξ_n^- in U_n which are contained in $D_n \subset U_n$. In view of Theorem 1.1, $\{\xi_n^+, \xi_n^-\} = \{\lambda_n^+, \lambda_n^-\}$, and the estimate of the periodic eigenvalues follows. To obtain the refined estimate of the gap lengths γ_n stated in Proposition 6.3 note that

$$|\gamma_n(\varphi)|^p = |\xi_n^+ - \xi_n^-|^p \leqslant 6^{p/2} |b_n^+ b_n^-|_{U_n}^{p/2} \leqslant 3^{p/2}(|b_n^+|_{U_n}^p + |b_n^-|_{U_n}^p).$$

Taking into account that $|b_n^\pm| \leqslant |\varphi_n^\pm| + |b_n^\pm - \varphi_n^\pm|$, it follows from the preceding line that

$$\frac{1}{2^p} \sum_{|n| \geqslant N} w_{2n}^p |\gamma_n|^p \leqslant \sum_{|n| \geqslant N} w_{2n}^p (|\varphi_{2n}^-|^p + |\varphi_{2n}^+|^p) + \sum_{|n| \geqslant N} w_{2n}^p (|b_n^- - \varphi_{2n}^-|^p + |b_n^+ - \varphi_{2n}^+|^p).$$

The claimed estimate of the gap lengths now follows from Lemma 6.8. ∎

Chapter 2

Liouville coordinates

7. Overview

Let us briefly recall from [23] the construction of actions and angles for a potential $\varphi \in L^2_r$. It follows from the Lax-pair formulation of the NLS equation, that the periodic spectrum of φ, defined as the spectrum of the Zakharov-Shabat operator $L(\varphi)$ endowed with periodic boundary conditions, is conserved by the NLS flow. The whole phase space L^2_r thus decomposes into *isospectral sets*

$$\operatorname{Iso}(\varphi) := \{\psi \in L^2_r : \operatorname{spec}_{per}(\varphi) = \operatorname{spec}_{per}(\psi)\},$$

which are invariant under the NLS flow. Moreover, the isospectral sets can be parametrized by the Dirichlet eigenvalues $(\mu_n)_{n\in\mathbb{Z}}$ of φ (which are *not* conserved) plus the signs $(\sigma_n)_{n\in\mathbb{Z}}$, where, roughly speaking, σ_n indicates the direction in which the Dirichlet eigenvalue μ_n is moving within the gap $G_n = [\lambda_n^-, \lambda_n^+]$ according to the NLS flow. It turns out that the isospectral set $\operatorname{Iso}(\varphi)$ is indeed a compact connected torus whose dimension equals the number of noncollapsed gaps G_n of φ – this number is infinite generically. The periodic spectrum is furthermore uniquely determined by the sequence of squared gap-lengths $\gamma_n^2(\varphi)$. These observations can be developed into the construction of action-angle variables $(I_n, \theta_n)_{n\in\mathbb{Z}}$, where the actions I_n are essentially given by γ_n^2 and play the role of squared radii of the torus $\operatorname{Iso}(\varphi)$, while the angles θ_n are defined in terms of the Dirichlet spectrum.

To give a more precise definition of the action-angle variables, we introduce another characterization of the periodic spectrum. Denote by $M(x, \lambda, \varphi)$ the standard fundamental solution of the ordinary differential equation $L(\varphi)M = \lambda M$ with $M(x = 0) = \mathrm{Id}$, and denote by $\Delta(\lambda, \varphi)$ the discriminant

$$\Delta(\lambda, \varphi) := \operatorname{tr} M(1, \lambda, \varphi).$$

To simplify matters, we may drop some or all of its arguments from the notation whenever there is no danger of confusion. The periodic spectrum of φ is precisely the zero set of the entire function $\Delta^2(\lambda) - 4$, which has the product representation

$$\Delta^2(\lambda) - 4 = -4 \prod_{n\in\mathbb{Z}} \frac{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}{\pi_n^2}, \qquad \pi_n := \begin{cases} n\pi, & n \neq 0, \\ 1, & n = 0. \end{cases} \tag{2.1}$$

The action variables of $\varphi \in L^2_r$ are now given by Arnold's formula

$$I_n = \frac{1}{\pi} \int_{\Gamma_n} \frac{\lambda \Delta^{\bullet}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} \,\mathrm{d}\lambda, \qquad n \in \mathbb{Z}. \tag{2.2}$$

Here Γ_n denotes a counterclockwise oriented circuit, which is circling sufficiently close around G_n, and $\sqrt[c]{\Delta^2(\lambda)-4}$ denotes the *canonical root*, which is defined on $\mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} G_n$, and has its sign determined by

$$\mathrm{i}\sqrt[c]{\Delta^2(\lambda)-4} > 0, \qquad \lambda_0^+ < \lambda < \lambda_1^-.$$

Proceeding from this definition, one can show that the action variables extend analytically to a common complex neighborhood of L_r^2 within L_c^2.

To define the angle variables, we also introduce the anti-discriminant δ as the anti-trace of the fundamental solution

$$\delta(\lambda,\varphi) \coloneqq \text{anti tr}\, M(1,\lambda,\varphi).$$

The angle variable θ_n of $\varphi \in L_r^2$ is defined, provided $\gamma_n \neq 0$, by

$$\theta_n = \left(\int_{\lambda_n^-}^{\mu_n} \frac{\psi_n(\lambda)}{\sqrt[*]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda \mod 2\pi \right) + \sum_{m\neq n} \int_{\lambda_m^-}^{\mu_m} \frac{\psi_n(\lambda)}{\sqrt[*]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda, \tag{2.3}$$

where each path of integration is chosen to not intersect the gap G_n except at its end points, and the sign of the root is chosen so that at the end point of the path $\sqrt[*]{\Delta^2(\lambda)-4}\,\big|_{\lambda=\mu_n} = \delta(\mu_n)$. Furthermore, the functions ψ_n are entire functions of the form

$$\psi_n(\lambda) = -\frac{2}{\pi_n} \prod_{m\neq n} \frac{\sigma_m^n - \lambda}{\pi_n}, \qquad \sigma_m^n = m\pi + \ell_m^2,$$

which are characterized by the conditions

$$\frac{1}{2\pi} \int_{\Gamma_m} \frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = \delta_{mn}, \qquad m,n \in \mathbb{Z}.$$

The angle variable θ_n is only well-defined mod 2π, and can be shown to be real analytic mod π on $L_r^2 \setminus Z_n$ where $Z_n = \{\psi \in L_r^2 : \gamma_n^2(\psi) = 0\}$ is a real analytic submanifold of L_r^2 of codimension one. These angles are canonically conjugated to the actions I_n in the sense that

$$\{I_n, I_m\} = 0 \text{ on } L_r^2, \qquad \{\theta_n, I_m\} = \delta_{nm} \text{ on } L_r^2 \setminus Z_n, \qquad \{\theta_n, \theta_m\} = 0 \text{ on } L_r^2 \setminus (Z_n \cup Z_m).$$

The purpose of this chapter is to extend the definition of the action-angle variables to the case where φ is in the Fourier Lebesgue space FL_r^p, $1 < p < \infty$. The main issue here is that for $p > 2$ the notion of the fundamental solution is not available. The standard existence and uniqueness theory for ordinary differential equations of the form $\partial_x M = A(x,\varphi)M$ is applicable for coefficients φ in L_c^1, while for any $p > 2$ the space FL_r^p contains elements which are not measures. To avoid these technical difficulties, we directly define the discriminant, for any φ in FL_c^p, in terms of an appropriate infinite product representation involving the periodic eigenvalues. The estimates of these eigenvalues obtained in the previous chapter ensure that this infinite product converges to an entire function on $\mathbb{C}\times FL_c^p$ which coincides, when restricted to $\mathbb{C}\times L_c^2$, with the classical discriminant. Extending the canonical root in a similar fashion then allows to define the actions by the well known formula (2.2). From this point, the construction of the functions ψ_n is straightforward and follows by the procedure laid out in [23] for the case of $\varphi \in L_c^2$. Defining also the anti-discriminant in terms of an infinite product ultimately allows to define the angle variables by formula (2.3). After establishing the analyticity of the action-angle coordinates, it follows by a density argument from the case $p = 2$ that they are canonically conjugated for any $1 < p < \infty$.

The main results of this chapter can be summarized as follows.

Theorem 7.1 *For any $1 < p < \infty$ there exists a neighborhood $\mathcal{W}^p$ of FL_r^p within FL_c^p so that*

(i) each action-variable I_n, $n \in \mathbb{Z}$, admits a real analytic extension to $\mathcal{W}^p$,

(ii) each angle-variable θ_n, $n \in \mathbb{Z}$, defined modulo 2π, is a real valued function on $FL_r^p \setminus Z_n$ and extends to a real analytic function on $\mathcal{W}^p \setminus Z_n$ when taken modulo π.

(iii) The action-angle variables are canonically conjugated in the sense that

$$\{I_n, I_m\} = 0 \text{ on } FL_r^p,$$
$$\{\theta_n, \theta_m\} = 0 \text{ on } FL_r^p \setminus (Z_n \cup Z_m),$$
$$\{\theta_n, I_m\} = \delta_{nm} \text{ on } FL_r^p \setminus Z_n. \quad \rtimes$$

8. Discriminant

In this section we extend the definition of the discriminant $\Delta(\lambda, \varphi)$ to the case where φ is an element of the Fourier Lebesgue space FL_c^p by using an appropriate representation of Δ as an infinite product involving the periodic spectrum investigated in the previous chapter. We proceed by deriving several estimates of this extension, and also prove the continuity of the periodic spectrum on FL_r^p.

For any $\varphi \in L_c^2$ according to [23] the entire function $\Delta^2(\lambda, \varphi) - 4$ admits the infinite product representation given in (2.1). Moreover, this function factors into $(\Delta(\lambda, \varphi) - 2)(\Delta(\lambda, \varphi) + 2)$ where $\Delta(\lambda, \varphi) - 2$ vanishes if and only if λ is a 1-periodic eigenvalue, while $\Delta(\lambda, \varphi) + 2$ vanishes if and only if λ is a 1-antiperiodic eigenvalue. The factors $\Delta(\lambda, \varphi) - 2$ and $\Delta(\lambda, \varphi) + 2$ can be shown to admit an analogous representation as an infinite product involving only the 1-periodic and 1-antiperiodic eigenvalues, respectively. If φ is of real type, then $\lambda_{2n}^\pm$ is 1-periodic while $\lambda_{2n+1}^\pm$ is anti-periodic. For a general potential $\varphi \in L_c^2$ not of real type, due to the lexicographical ordering of the eigenvalues, it is no longer true that $\lambda_{2n}^\pm$ ($\lambda_{2n+1}^\pm$) is 1-periodic (1-antiperiodic). However, one can choose $N \geqslant 1$ locally uniformly on L_c^2 so that the circle of radius $(2N + 1/2)\pi$ contains precisely $4N + 2$ 1-periodic and $4N$ 1-antiperiodic eigenvalues while for any $|n| > N$ we have that $\lambda_{2n}^\pm$ is 1-periodic while $\lambda_{2n+1}^\pm$ is 1-antiperiodic. Consequently, the sequence of 1-periodic eigenvalues, counted with multiplicities and denoted by $(\lambda_k^p)_{k \in \mathbb{Z}}$, can be ordered lexicographically so that

$$\cdots \preccurlyeq \lambda_{2n-1}^p \preccurlyeq \lambda_{2n}^p \preccurlyeq \lambda_{2n+1}^p \preccurlyeq \lambda_{2n+2}^p \preccurlyeq \cdots, \qquad \lambda_{2n}^p, \lambda_{2n+1}^p = 2n\pi + \ell_n^2.$$

Clearly, $\lambda_{2n}^p = \lambda_{2n}^-$ and $\lambda_{2n+1}^p = \lambda_{2n}^+$ for all $|n|$ sufficiently large. Similarly, the sequence of 1-antiperiodic eigenvalues, counted with multiplicities and denoted by $(\lambda_k^a)_{k \in \mathbb{Z}}$, can be ordered lexicographically so that

$$\cdots \preccurlyeq \lambda_{2n-1}^a \preccurlyeq \lambda_{2n}^a \preccurlyeq \lambda_{2n+1}^a \preccurlyeq \lambda_{2n+2}^a \preccurlyeq \cdots, \qquad \lambda_{2n}^a, \lambda_{2n+1}^a = (2n+1)\pi + \ell_n^2,$$

where $\lambda_{2n}^a = \lambda_{2n+1}^-$ and $\lambda_{2n+1}^a = \lambda_{2n+1}^+$ for all $|n|$ sufficiently large. The product representation of the discriminant is now given by – c.f. [20]

$$\begin{aligned}\Delta(\lambda,\varphi)-2&=-2\prod_{m\in\mathbb{Z}}\frac{(\lambda_{2m}^{p}-\lambda)(\lambda_{2m+1}^{p}-\lambda)}{\pi_{2m}^{2}},\\ \Delta(\lambda,\varphi)+2&=2\prod_{m\in\mathbb{Z}}\frac{(\lambda_{2m}^{a}-\lambda)(\lambda_{2m+1}^{a}-\lambda)}{\pi_{2m+1}^{2}}.\end{aligned}\tag{2.4}$$

By Theorem 1.1 any potential $\varphi\in FL_c^p$, $1<p<\infty$ admits bi-infinite sequences of 1-periodic and 1-antiperiodic eigenvalues which can be ordered lexicographically so that

$$\begin{aligned}&\cdots\preccurlyeq\lambda_{2n-1}^{p}\preccurlyeq\lambda_{2n}^{p}\preccurlyeq\lambda_{2n+1}^{p}\preccurlyeq\lambda_{2n+2}^{p}\preccurlyeq\cdots,\qquad \lambda_{2n}^{p},\lambda_{2n+1}^{p}=2n\pi+\ell_n^p,\\ &\cdots\preccurlyeq\lambda_{2n-1}^{a}\preccurlyeq\lambda_{2n}^{a}\preccurlyeq\lambda_{2n+1}^{a}\preccurlyeq\lambda_{2n+2}^{a}\preccurlyeq\cdots,\qquad \lambda_{2n}^{a},\lambda_{2n+1}^{a}=(2n+1)\pi+\ell_n^p.\end{aligned}$$

We use the formulas (2.4) to extend the definition of the discriminant to the case where $\varphi\in FL_c^p$.

Lemma 8.1 *For any $1<p<\infty$,*

(i) the functions

$$\begin{aligned}f(\lambda,\varphi)&=-2\prod_{m\in\mathbb{Z}}\frac{(\lambda_{2m}^{p}-\lambda)(\lambda_{2m+1}^{p}-\lambda)}{\pi_{2m}^{2}},\\ g(\lambda,\varphi)&=2\prod_{m\in\mathbb{Z}}\frac{(\lambda_{2m}^{a}-\lambda)(\lambda_{2m+1}^{a}-\lambda)}{\pi_{2m+1}^{2}},\end{aligned}$$

are analytic on on $\mathbb{C}\times FL_c^p$.

(ii) $\Delta(\lambda,\varphi)=f(\lambda,\varphi)+2=g(\lambda,\varphi)-2$ is the unique analytic extension of the discriminant to $\mathbb{C}\times FL_c^p$.

(iii) locally uniformly on FL_c^p and uniformly in $\lambda\notin\Pi:=\bigcup_{n\in\mathbb{Z}}D_n$ as $|\lambda|\to\infty$

$$\begin{aligned}\Delta(\lambda,\varphi)&=(2\cos\lambda)(1+o(1)),\\ \Delta^{\bullet}(\lambda,\varphi)&=(-2\sin\lambda)(1+o(1)).\end{aligned}$$

In more detail, for every $\varepsilon>0$ and $\varphi\in FL_c^p$ there exists a neighborhood V of φ and $\Lambda>0$ so that

$$\sup_{\substack{\lambda\notin\Pi\\|\lambda|>\Lambda}}\left|\frac{\Delta(\lambda,\psi)}{2\cos\lambda}-1\right|\leqslant\varepsilon,\qquad \psi\in V.$$

(iv) the periodic spectrum of $\varphi\in FL_c^p$ is precisely the zero set of the entire function $\Delta^2(\lambda)-4$ which admits the product representation

$$\Delta^2(\lambda,\varphi)-4=f(\lambda,\varphi)g(\lambda,\varphi)=-4\prod_{m\in\mathbb{Z}}\frac{(\lambda_m^+-\lambda)(\lambda_m^--\lambda)}{\pi_m^2}.\qquad\rtimes$$

Proof. (i) By Lemma D.5 and Theorem 1.2

$$f_n(\lambda,\varphi)=-\prod_{|m|\leqslant n}\frac{(\lambda_{2m}^{p}-\lambda)(\lambda_{2m+1}^{p}-\lambda)}{\pi_{2m}^{2}}$$

converges locally uniformly to f on $\mathbb{C}\times FL_c^p$ and f satisfies on $\mathbb{C}\setminus\Pi$

$$\begin{aligned}f(\lambda,\varphi)&=-4\sin^2(\lambda/2)(1+o(1))\\ &=(2\cos\lambda-2)(1+o(1))=(2\cos\lambda)(1+o(1))-2.\end{aligned}$$

To prove analyticity in φ, choose an integer $N_\varphi \geqslant 1$ and an open neighborhood U_φ of φ within FL_c^p according to Theorem 3.1. For each fixed $n \geqslant N_\varphi$, the projector

$$\Pi_n^+(\psi) = \frac{1}{2\pi \mathrm{i}} \int_{C_n} (\lambda - L(\psi))^{-1} \, \mathrm{d}\lambda \Big|_{FL^p_{per+}},$$

with $C_n = \{\lambda \in \mathbb{C} : |\lambda| = (n + 1/2)\pi\}$, is analytic on U_φ. Its range $\mathcal{P}_n^+(\psi) := \Pi_n^+(\psi)(FL^p_{per+})$ is finite dimensional, and we have for f_n the identity

$$f_n(\lambda, \psi) = \left(\prod_{|m| \leqslant n} \frac{1}{\pi^2_{2m}} \right) \det\left(L(\psi)\Big|_{\mathcal{P}_n^+(\psi)} - \lambda \mathrm{Id}\Big|_{\mathcal{P}_n^+(\psi)} \right).$$

After possibly shrinking U_φ, the ranges of $\mathcal{P}_n^+(\varphi)$ and $\mathcal{P}_n^+(\psi)$ are isomorphic for all $\psi \in U_\varphi$ and the isomorphism $T_\psi \colon \mathcal{P}_n(\varphi) \to \mathcal{P}_n(\psi)$ depends analytically on ψ – see [40, Remark 4.4]. Consequently,

$$f_n(\lambda, \psi) = \left(\prod_{|m| \leqslant n} \frac{1}{\pi^2_{2m}} \right) \det\left(T_\psi^{-1} L(\psi) T_\psi\Big|_{\mathcal{P}_n^+(\varphi)} - \lambda \mathrm{Id}\Big|_{\mathcal{P}_n^+(\varphi)} \right).$$

This shows the analyticity of f_n on $\mathbb{C} \times U_\varphi$. Since f is the locally uniform limit of f_n the claimed analyticity of f follows. The proof for g is similar.

(ii) The identity $\Delta = f + 2 = g - 2$ holds on L_c^2 and extends by continuity to FL_c^p.

(iii) The estimate of Δ follows from the one of f. The estimate of $\Delta^\bullet$ follows from Cauchy's estimate and the fact that $|\frac{\cos\lambda}{\sin\lambda}| \leqslant \frac{\mathrm{e}^{|\Im\lambda|}}{|\sin\lambda|} \leqslant 4$ for $\lambda \notin \Pi$ – c.f. [23, Lemma 6.2].

(iv) In view of Lemma D.5 the zero set of the product $f(\lambda, \varphi) g(\lambda, \varphi)$ is precisely the periodic spectrum of φ counted with multiplicities. The product representation follows from the ones of f and g. ∎

We are now in a position to prove the continuity of the periodic eigenvalues on FL_r^p. Note that for a general potential in FL_c^p the periodic eigenvalues are not continuous due to the lexicographical ordering.

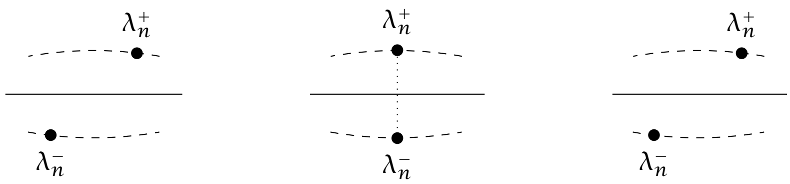

Figure 2.1.: Discontinuity of the periodic eigenvalues for a general potential.

Lemma 8.2 *Suppose $1 < p < \infty$.*

(i) At each point $\varphi \in FL_r^p$ the functions $\psi \mapsto \lambda_n^\pm(\psi)$, $n \in \mathbb{Z}$, are continuous.

(ii) On FL_r^p

$$\cdots \leqslant \lambda_{n-1}^+ < \lambda_n^- \leqslant \lambda_n^+ < \lambda_{n+1}^- \leqslant \cdots. \quad \rtimes$$

Proof. (i) Fix $\varphi \in FL_r^p$ and any $n > N_\varphi$. For each eigenvalue $\lambda_m^\pm$, $|m| \leqslant n$, let $U_m^\pm$ denote a closed disc of radius ε centered at $\lambda_m^\pm$. If we choose $\varepsilon > 0$ small enough, then any two discs are either disjoint or identical (in case the corresponding eigenvalues are have the same position). The set $D = \bigcup_{|m| \leqslant n} U_m^- \cup U_m^+$ is compact and $\Delta^2(\lambda, \varphi) - 4$ does not vanish on ∂D. It follows from Rouche's theorem, provided $\|\varphi - \psi\|_p$ is sufficiently small, that $\Delta^2(\lambda,\varphi) - 4$ and $\Delta^2(\lambda,\psi) - 4$ have the same number of roots, counted with multiplicity in each disc $U_m^{\pm}$, $|m| \leqslant n$. Moreover, it follows from the lexicographical ordering that $\lambda_m^{\pm}(\psi)$ is contained in $U_m^{\pm}$, that is $|\lambda_m^{\pm}(\varphi) - \lambda_m^{\pm}(\psi)| < \varepsilon$ which gives the claim.

(ii) If φ is of real type, then the periodic spectrum is real and the lexicographical ordering reduces to the ordering of real numbers

$$\cdots \leqslant \lambda_{n-1}^+ \leqslant \lambda_n^- \leqslant \lambda_n^+ \leqslant \lambda_{n+1}^- \leqslant \cdots .$$

Hence it is to show that distinct gaps are disjoint, that is $\lambda_{n-1}^+ \neq \lambda_n^-$. For any $\varphi \in L_r^2$ we have

$$(-1)^n \Delta(\lambda_n^{\pm}) = 2, \qquad n \in \mathbb{Z}, \tag{2.5}$$

and by the continuity of the periodic spectrum this identity extends to FL_r^p. In particular, $\lambda_{n-1}^+ \neq \lambda_n^-$ for all $n \in \mathbb{Z}$. ∎

So for $\varphi \in FL_r^p$ all periodic eigenvalues come in well separated pairs, forming the gaps

$$G_n = [\lambda_n^-, \lambda_n^+], \qquad n \in \mathbb{Z}.$$

Moreover, for any $\lambda \in G_n$, one has

$$(-1)^n \Delta(\lambda) \geqslant 2.$$

There can be no other points on the real line where this condition is true, since this would imply the existence of additional periodic eigenvalues in view of the mean value theorem. Therefore, we obtain the following characterization of the gaps, as subsets of the real line,

$$|\Delta(\lambda)| \geqslant 2 \Leftrightarrow \lambda \in \bigcup_{n \in \mathbb{Z}} G_n.$$

We proceed by investigating the λ-derivative $\Delta^{\bullet}$ of the discriminant and its roots.

Lemma 8.3 *For $\varphi \in FL_c^p$, $1 < p < \infty$, and $N > 0$ sufficiently large, the function $\Delta^{\bullet}$ has one root $\lambda_n^{\bullet}$ in each disc D_n, $|n| > N$, and $2N + 1$ roots in the disc B_N; there are no other roots. Here, N can be chosen locally uniformly in φ. If φ is of real type, then the roots are real.* ⋊

Proof. Since $\Delta^{\bullet}(\lambda) = (-2\sin\lambda)(1 + o(1))$ as $|\lambda| \to \infty$ on $\mathbb{C} \setminus \Pi$ locally uniformly in φ by Lemma 8.1, it follows from Rouche's Theorem that each disc D_n for $|n| > N$ sufficiently large contains precisely one root, while $C_N = \{\lambda \in \mathbb{C} : |\lambda| \leqslant (N + 1/2)\pi\}$ contains $2N + 1$ roots when counted with multiplicity. Applying Rouche's Theorem on C_{N+k}, $k \geqslant 1$, shows that there are no other roots.

Suppose φ is of real type, then $(-1)^n \Delta(\lambda) \geqslant 2$ on G_n and $(-1)^n \Delta(\lambda) < 2$ on $(\lambda_n^+, \lambda_{n-1}^-)$. Consequently, Δ has an extremum in each gap G_n, and it follows by a simple counting argument that all roots of $\Delta^{\bullet}$ are such extrema. ∎

The previous lemma shows $\lambda_n^{\bullet} = n\pi + O(1)$ locally uniformly in φ. To improve the asymptotics of the roots of $\Delta^{\bullet}$ we need the following estimate.

Lemma 8.4 *For any potential in FL_c^p, $1 < p < \infty$ and any sequence λ_n with $\lambda_n \in D_n$,*

$$\Delta(\lambda_n) = 2\cos\lambda_n + \ell_n^p, \qquad \Delta^\bullet(\lambda_n) = -2\sin\lambda_n + \ell_n^p,$$

locally uniformly on FL_c^p. ⋊

Proof. The claim for Δ follows from Lemma D.9 and the claim for $\Delta^\bullet$ follows from Cauchy's estimate. ∎

Lemma 8.5 *At each potential in FL_c^p, $1 < p < \infty$, the roots of the function $\Delta^\bullet$ form a bi-infinite sequence $\cdots \preccurlyeq \lambda_{n-1}^\bullet \preccurlyeq \lambda_n^\bullet \preccurlyeq \lambda_{n+1}^\bullet \preccurlyeq \cdots$ so that*

$$\lambda_n^\bullet = n\pi + \ell_n^p, \qquad \Delta^\bullet(\lambda, \varphi) = 2\prod_{m\in\mathbb{Z}} \frac{\lambda_m^\bullet - \lambda}{\pi_m}, \tag{2.6}$$

where the estimate is locally uniform in φ. Moreover, each function $\psi \mapsto \lambda_n^\bullet(\psi)$ is continuous at any point $\varphi \in FL_r^p$. ⋊

Proof. By the estimate of Lemma 8.4

$$0 = \Delta^\bullet(\lambda_n^\bullet) = \sin\lambda_n^\bullet + \ell_n^p,$$

hence $\sin\lambda_n^\bullet = \ell_n^p$. By Lemma 8.3, there exists $N \geqslant 1$ such that $|\lambda_n^\bullet - n\pi| \leqslant \pi/4$ for $|n| > N$ thus $\lambda_n^\bullet = n\pi + \ell_n^p$. Since N can be chosen locally uniformly in φ and $\sum_{|n|\leqslant N} |\lambda_n^\bullet - n\pi|^p$ is locally uniformly bounded, we conclude the estimate of $\lambda_n^\bullet$ is locally uniform in φ.

As a consequence, by Lemma D.5 the infinite product

$$\phi(\lambda) = 2\prod_{m\in\mathbb{Z}} \frac{\lambda_m^\bullet - \lambda}{\pi_m}$$

is a well defined entire function with roots $\lambda_m^\bullet$, $m \in \mathbb{Z}$. The quotient of ϕ and $\Delta^\bullet$ is therefore an entire function as well. Moreover, by Lemma D.5 the asymptotic behavior of ϕ is $\phi(\lambda) = (-2\sin\lambda)(1 + o(1))$ uniformly on $\mathbb{C} \setminus \Pi$ as $|\lambda| \to \infty$, hence in view of Lemma 8.1 the quotient converges uniformly to 1 as $|\lambda| \to \infty$ which proves that $\phi = \Delta^\bullet$.

The continuity of the roots $\lambda_n^\bullet$ on FL_r^p is proved using Rouche's Theorem in analogous fashion to the proof of Lemma 8.2. ∎

It follows by continuity that $\lambda_n^- \leqslant \lambda_n^\bullet \leqslant \lambda_n^+$ for any $\varphi \in FL_r^p$. This shows that for real type potentials Δ attains a single strict extremum within each gap G_n and has no other critical points on the real line.

The following result is an adaption of [23, Lemma 6.9] and improves on the asymptotics of $\lambda_n^\bullet$ stated in Lemma 8.5.

Lemma 8.6 *Locally uniformly on FL_c^p, $1 < p < \infty$, for $|n|$ sufficiently large*

$$\lambda_n^\bullet - \tau_n = \gamma_n^2 \ell_n^p.$$ ⋊

Proof. Write $\Delta^2(\lambda) - 4 = 4((\tau_n - \lambda)^2 - \gamma_n^2/4)\Delta_n(\lambda)$ where

$$\Delta_n(\lambda) = \frac{1}{\pi_n^2}\prod_{m\neq n} \frac{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}{\pi_m^2}.$$

By Lemma D.8 and the fact that $\frac{\sin(\lambda-n\pi)}{\lambda-n\pi} = 1 + O(\lambda - n\pi)$ uniformly on D_n

$$\sup_{\lambda\in D_n}\left|\Delta_n(\lambda) - \left(\frac{\sin(\lambda-n\pi)}{\lambda-n\pi}\right)^2\right| = \ell_n^p.$$

Hence from $\lambda_n^\bullet = n\pi + \ell_n^p$ we conclude $\Delta_n(\lambda_n^\bullet) = 1 + \ell_n^p$. Moreover, by Cauchy's estimate

$$\partial_\lambda\left(\Delta_n(\lambda) - \left(\frac{\sin(\lambda-n\pi)}{\lambda-n\pi}\right)^2\right)\Big|_{\lambda_n^\bullet} = \ell_n^p$$

Since $\frac{\sin(\lambda-n\pi)}{\lambda-n\pi} = \int_0^1 \cos(s(\lambda-n\pi))\,\mathrm{d}s$, we conclude $\partial_\lambda\left(\frac{\sin(\lambda-n\pi)}{\lambda-n\pi}\right)^2 = O(\lambda-n\pi)$ and thus get $\Delta_n^\bullet(\lambda_n^\bullet) = \ell_n^p$. Consequently,

$$0 = \frac{1}{4}(\Delta^2(\lambda)-4)\Big|_{\lambda_n^\bullet} = 2(\lambda_n^\bullet-\tau_n)\Delta_n(\lambda_n^\bullet) + \left((\lambda_n^\bullet-\tau_n)^2 - \gamma_n^2/4\right)\Delta_n^\bullet(\lambda_n^\bullet),$$

and hence, since $\lambda_n^\bullet - \tau_n = O(1)$,

$$(1+\ell_n^p)(\lambda_n^\bullet - \tau_n) = \gamma_n^2\ell_n^p.$$

For all $|n|$ sufficiently large, the prefactor on the left hand side is uniformly bounded away from zero so that $\lambda_n^\bullet - \tau_n = \gamma_n^2\ell_n^p$. ∎

9. Characteristic functions for the Dirichlet and Neumann spectra

In this section we extend the definition of the characteristic functions of the Dirichlet and Neumann spectra as well as the anti-discriminant to the case where $\varphi \in FL_c^p$.

Denote by $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ the fundamental solution of $\varphi \in L_c^2$. The Dirichlet spectrum $(\mu_n)_{n\in\mathbb{Z}}$ is precisely the zero set of the entire function

$$\chi_D = \frac{\grave{m}_4 + \grave{m}_3 - \grave{m}_2 - \grave{m}_1}{2\mathrm{i}}$$

– see [23, Section 5]. Moreover, χ_D admits the following product representation

$$\chi_D(\lambda,\varphi) = -\prod_{m\in\mathbb{Z}}\frac{\mu_m - \lambda}{\pi_m}.$$

Similarly, the Neumann spectrum $(\nu_n)_{n\in\mathbb{Z}}$ is the zero set of the entire function

$$\chi_N(\lambda,\varphi) = \frac{\grave{m}_4 - \grave{m}_3 + \grave{m}_2 - \grave{m}_1}{2\mathrm{i}},$$

which admits the product representation

$$\chi_N(\lambda,\varphi) = -\prod_{m\in\mathbb{Z}}\frac{\nu_m - \lambda}{\pi_m}.$$

We use these product representations and the asymptotic behavior of the Dirichlet and Neumann spectra obtained in the previous chapter to extend the functions χ_D and χ_N to the case where $\varphi \in FL_c^p$.

Lemma 9.1 *Suppose* $1 < p < \infty$.

(i) The functions

$$h_D(\lambda, \varphi) = -\prod_{m\in\mathbb{Z}} \frac{\mu_m - \lambda}{\pi_m}$$

$$h_N(\lambda, \varphi) = -\prod_{m\in\mathbb{Z}} \frac{\nu_m - \lambda}{\pi_m}$$

are analytic functions on $\mathbb{C} \times FL_c^p$ *whose zero sets are precisely the Dirichlet and Neumann spectra, respectively.*

(ii) At each point $\varphi \in FL_r^p$ *the functions*

$$\psi \mapsto \mu_n(\psi), \quad \psi \mapsto \nu_n(\psi), \quad n \in \mathbb{Z},$$

are continuous.

(iii) On FL_r^p

$$\cdots \leqslant \lambda_{n-1}^+ < \lambda_n^- \leqslant \mu_n, \nu_n \leqslant \lambda_n^+ < \lambda_{n+1}^- \leqslant \cdots . \quad \rtimes$$

Proof. The proof of claim (i) is completely analogous to the proof of Lemma 8.1 and is therefore omitted. The continuity stated in item (ii) follows from Rouche's Theorem analogously to the proof of Lemma 8.2. Finally, we note that on L_r^2 one has $(-1)^n \Delta(\mu_n) \geqslant 2$ for any $n \in \mathbb{Z}$ and by continuity this identity extends to FL_r^p, which shows that $\mu_n \in G_n$ as claimed in (iii). The proof for ν_n is similar. ∎

Mutantis mutandis one obtains the same results for the auxiliary D and N spectra, whose characteristic functions on L_c^2 are given by

$$\chi_D^*(\lambda, \varphi) = \frac{\dot{m}_4 + \mathrm{i}\dot{m}_3 + \mathrm{i}\dot{m}_2 - \dot{m}_1}{2\mathrm{i}} = -\prod_{m\in\mathbb{Z}} \frac{\mu_m^* - \lambda}{\pi_m},$$

$$\chi_N^*(\lambda, \varphi) = \frac{\dot{m}_4 - \mathrm{i}\dot{m}_3 - \mathrm{i}\dot{m}_2 - \dot{m}_1}{2\mathrm{i}} = -\prod_{m\in\mathbb{Z}} \frac{\nu_m^* - \lambda}{\pi_m}.$$

In particular, the anti-discriminant may be written as

$$\delta = \dot{m}_2 + \dot{m}_3 = \chi_D^* - \chi_N^*.$$

Lemma 9.2 *Suppose* $1 < p < \infty$.

(i) The anti-discriminant $\delta(\lambda, \varphi)$ *admits an analytic extension to* $\mathbb{C} \times FL_c^p$.

(ii) At any simple Dirichlet eigenvalue μ_n *one has*

$$\Delta^2(\mu_n) - 4 = \delta(\mu_n)^2.$$

(iii) Locally uniformly on FL_c^p

$$\delta(\mu_n) = \ell_n^p, \qquad \dot{\delta}(\mu_n) = \ell_n^p.$$

Proof. (i) Since χ_D^* and χ_N^* admit an analytic extension to $\mathbb{C} \times FL_c^p$ in terms of their product representation, the analytic extension of the anti-discriminant is given by $\delta = \chi_D^* - \chi_N^*$.

(ii) The identity $\Delta^2(\mu_n) - 4 = \delta(\mu_n)^2$ holds on L_c^2 for any Dirichlet eigenvalue. If the Dirichlet eigenvalue is simple at φ, it is a simple root of χ_D and hence by the implicit function theorem defines an analytic function locally around φ. The claimed identity thus follows by density and continuity.

(iii) By Lemma D.9 we have $\chi_D^*(\lambda,\varphi) - \sin\lambda = \ell_n^p$ and $\chi_N^*(\lambda,\varphi) - \sin\lambda = \ell_n^p$ uniformly on D_n hence

$$\delta(\lambda) = \chi_D^*(\lambda,\varphi) - \chi_N^*(\lambda,\varphi) = \ell_n^p$$

uniformly on D_n. The second estimate follows by Cauchy's estimate. ∎

10. Potentials of almost real type

To obtain several analyticity results we need to consider potentials in some small complex neighborhood of FL_r^p. In this section we choose this neighborhood in such a way that the spectra of potentials of that neighborhood are isolated in the following sense: We say that a sequence $(U_n)_{n\in\mathbb{Z}}$ of complex discs forms a set of isolating neighborhoods for a potential φ in FL_c^p if it has the following properties

(i) $G_n \subset U_n$ and $\mu_n, \nu_n, \lambda_n^\bullet \in U_n$ for any $n \in \mathbb{Z}$.

(ii) There exists a constant $c \geqslant 1$ such that for all $m \neq n$

$$c^{-1}|m-n| \leqslant \operatorname{dist}(U_m, U_n) \leqslant c|m-n|. \tag{2.7}$$

(iii) For $|n|$ sufficiently large, $U_n = D_n$.

Lemma 10.1 *For any $1 < p < \infty$ there exists an open an connected neighborhood $\hat{\mathcal{W}}^p$ of FL_c^p with the property that each potential $\varphi \in \hat{\mathcal{W}}^p$ admits an open and connected neighborhood V_φ, such that there exists a common sequence $(U_n)_{n\in\mathbb{Z}}$ of isolating neighborhoods centered on the real line for all potentials in V_φ.* ⋊

Throughout this text we denote by V_φ an open neighborhood of $\varphi \in \hat{\mathcal{W}}^p$ so that a common set of isolating neighborhoods $(U_n)_{n\in\mathbb{Z}}$ exists.

Proof. By Theorem 1.2, Theorem 1.5, and Lemma 8.5 for each potential $\varphi \in FL_r^p$ there exists an integer $N_\varphi \geqslant 0$ and an open neighborhood V_φ such that for all $|n| > N_\varphi$

$$G_n(\psi) \subset U_n, \qquad \mu_n(\psi), \nu_n(\psi), \lambda_n^\bullet(\psi) \in U_n, \qquad \psi \in V_\varphi, \tag{2.8}$$

with $U_n = D_n$ for $|n| > N_\varphi$. Since $\lambda_n^\pm$, $\lambda_n^\bullet$, μ_n, and ν_n are continuous at φ, and

$$\dots \lambda_{n-1}^+ < \lambda_n^- \leqslant \lambda_n^\bullet, \mu_n, \nu_n \leqslant \lambda_n^+ < \lambda_{n+1}^- \leqslant \dots,$$

after possibly shrinking V_φ, there exist mutually disjoint discs U_n, $|n| \leqslant N$, centered at the real line, such that (2.8) also holds for $|n| \leqslant N_\varphi$. Finally, we set $\hat{\mathcal{W}}^p = \bigcup_{\varphi\in FL_r^p} V_\varphi$. ∎

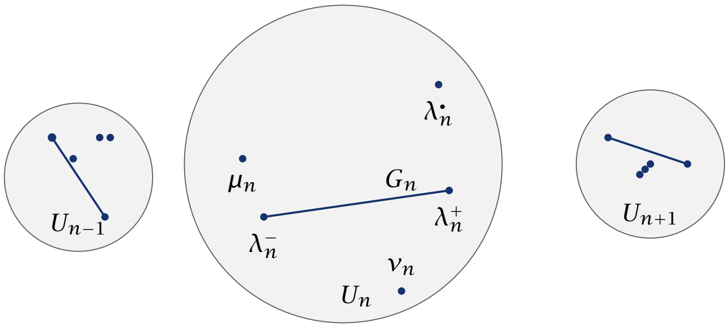

Figure 2.2.: Isolating neighborhoods U_n.

As an immediate consequence of this lemma, all Dirichlet and Neumann eigenvalues are simple and all $\lambda_n^\bullet$, $n \in \mathbb{Z}$, are simple roots of $\Delta^\bullet$ on $\hat{\mathcal{W}}^p$. By the implicit function theorem, they are therefore real analytic functions on $\hat{\mathcal{W}}^p$. The periodic eigenvalues, in contrast, are not even continuous on $\hat{\mathcal{W}}^p$ due to the lexicographical ordering. However, certain symmetric expressions involving the periodic eigenvalues are indeed continuos and even real analytic on $\hat{\mathcal{W}}^p$.

Lemma 10.2 *Suppose* $1 < p < \infty$.

(i) For each $k \geqslant 1$ *the functions*

$$(\lambda_n^+)^k + (\lambda_n^-)^k, \qquad n \in \mathbb{Z},$$

are real analytic on $\hat{\mathcal{W}}^p$.

(ii) Consequently, the functions

$$\tau_n = (\lambda_n^+ + \lambda_n^-)/2, \qquad \gamma_n^2 = (\lambda_n^+ - \lambda_n^-)^2, \qquad n \in \mathbb{Z},$$

are real analytic on $\hat{\mathcal{W}}^p$ *with asymptotics*

$$\tau_n = n\pi + \ell_n^p, \qquad \gamma_n^2 = \ell_n^{p/2},$$

locally uniformly on $\hat{\mathcal{W}}^p$.

(iii) Moreover,

$$(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda) = (\tau_n - \lambda)^2 - \gamma_n^2/4$$

is a real analytic function on $\mathbb{C} \times \hat{\mathcal{W}}^p$. ⋊

Proof. (i): The proof of the analyticity is analogous to the proof of [23, Lemma 12.3]. For any $\psi \in V_\varphi$ the only roots of $\Delta^2(\lambda) - 4$ which are contained in U_n are λ_n^+ and λ_n^-. Therefore, by the argument principle for any $k \geqslant 1$

$$(\lambda_n^+)^k + (\lambda_n^-)^k = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_n} \frac{2\Delta(\lambda)\Delta^\bullet(\lambda)}{\Delta^2(\lambda) - 4} \lambda^k \,\mathrm{d}\lambda.$$

Since $\Delta^2(\lambda) - 4$ does not vanish on $\partial U_n \times V_\varphi$, the right hand side is analytic on V_φ.

(ii): By writing $\tau_n = (\lambda_n^+ + \lambda_n^-)/2$ and $\gamma_n^2 = 2(\lambda_n^+)^2 + 2(\lambda_n^-)^2 - (\lambda_n^+ + \lambda_n^-)^2$, the analyticity follows from item (i). The asymptotic behavior has been proved in Theorem 1.2.

(iii): The identity $(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda) = (\tau_n - \lambda)^2 - \gamma_n^2/4$ follows from a straightforward computation and the right hand side is real analytic on $\mathbb{C} \times \hat{\mathcal{W}}^p$. ∎

It is convenient to denote by $\sqrt[+]{}$ the principal branch of the square root which is analytic on the complex plane minus the ray $(-\infty, 0]$. Moreover, we define the *standard roots*

$$w_n(\lambda, \varphi) = \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}, \qquad \lambda \notin G_n, \qquad n \in \mathbb{Z},$$

by the condition

$$w_n(\lambda, \varphi) \coloneqq (\tau_n - \lambda) \sqrt[+]{1 - \gamma_n^2/4(\tau_n - \lambda)^2}, \qquad \lambda \notin G_n. \tag{2.9}$$

Lemma 10.3 *For any $\varphi \in \hat{\mathcal{W}}^p$, $1 < p < \infty$, and $n \in \mathbb{Z}$, the standard root w_n is analytic in λ on $\mathbb{C} \setminus G_n$ and analytic in both variables on $(\mathbb{C} \setminus \overline{U_n}) \times V_\varphi$. Moreover, there exists a constant $c \geqslant 1$ such that for any $m \neq n$*

$$c^{-1}|m - n| \leqslant |w_n(\lambda, \psi)| \leqslant c|m - n|, \qquad (\lambda, \psi) \in U_m \times V_\varphi. \tag{2.10}$$

Finally,

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_m} \frac{\mathrm{d}\lambda}{w_n(\lambda)} = -\delta_{m,n}, \qquad m \in \mathbb{Z},$$

for any counterclockwise oriented contour Γ_m around G_m inside U_m. ⋊

Proof. The claimed analyticity of w_n follows from the analyticity of τ_n and γ_n^2 on V_φ and the fact that $1-\gamma_n^2/4(\tau_n-\lambda)^2 \in \mathbb{C}\setminus(-\infty, 0]$ for all $\lambda \notin G_n$. The estimate (2.10) is obtained from estimate (2.7) of the isolating neighborhoods and the representation $|w_n(\lambda)| = \sqrt[+]{|\lambda_n^+ - \lambda||\lambda_n^- - \lambda|}$. Finally, the integral identity follows by continuity from the case $p = 2$ considered in [23, Lemma 12.6]. ∎

If $\gamma_n = 0$, then $w_n(\lambda) = (\tau_n - \lambda)$ is analytic on U_n. On the other hand, if $\gamma_n \neq 0$, the standard root w_n extends continuously to the sides $G_n^\pm$ of G_n defined by

$$G_n^\pm \coloneqq \{\lambda_t^\pm \in \mathbb{C} : -1 \leqslant t \leqslant 1\}, \qquad \lambda_t^\pm \coloneqq \tau_n + (t \pm \mathrm{i}0)\gamma_n/2, \tag{2.11}$$

and admits opposite signs on opposite sides of $G_n^\pm$. Indeed

$$w_n(\lambda_t^\pm) = \mp \mathrm{i}\frac{\gamma_n}{2} \sqrt[+]{1 - t^2}, \qquad -1 \leqslant t \leqslant 1. \tag{2.12}$$

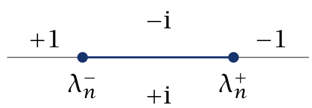

Figure 2.3.: Signs of $w_n(\lambda)$ around G_n.

Lemma 10.4 *Suppose $\gamma_n \neq 0$ and f is continuous on G_n, then*

$$\sup_{\lambda \in G_n^+ \cup G_n^-} \left| \frac{1}{\pi} \int_{\lambda_n^-}^{\lambda} \frac{f(z)}{w_m(z)} \, \mathrm{d}z \right| \leqslant \max_{\lambda \in G_n} |f(\lambda)|. \quad ⋊$$

Proof. We choose the parametrization $\lambda_t^\pm$ of $G_n^\pm$ to obtain for $-1 \leqslant t \leqslant 1$,

$$\int_{\lambda_n^-}^{\lambda_t^\pm} \frac{f(z)}{w_m(z)} \, \mathrm{d}z = \pm \mathrm{i} \int_{-1}^{t} \frac{f(\lambda_r^\pm)}{\sqrt[+]{1 - r^2}} \, \mathrm{d}r.$$

Since $\int_{-1}^{1} \frac{1}{\sqrt[+]{1-r^2}} \, \mathrm{d}r = \pi$, the claim follows immediately. ∎

A significant part of this work is concerned with infinite products involving the standard roots. Our first result is the following adaption of [23, Lemma 12.7].

Lemma 10.5 *For any φ in $\hat{\mathcal{W}}^p$, $1 < p < \infty$, and $n \in \mathbb{Z}$,*

$$f_n(\lambda) = \frac{1}{\pi_n} \prod_{m \neq n} \frac{w_m(\lambda)}{\pi_m}$$

defines a function which is analytic in λ on $\mathbb{C} \setminus \bigcup_{m \neq n} G_m$ and analytic in both variables on $(\mathbb{C} \setminus \bigcup_{m \neq n} \overline{U_m}) \times V_\varphi$. Moreover, f_n does not vanish on these domains. ⋊

Proof. For the sake of brevity we only consider the case $n = 0$. The proof can be adapted easily to the case of any $n \neq 0$. Let

$$a_k = \frac{w_k(\lambda)}{\pi_k} - 1 = \frac{w_k(\lambda) - \pi_k}{\pi_k}, \qquad k \neq 0.$$

Then each a_k is bounded in λ on compact subsets of $\mathbb{C} \setminus \bigcup_{m \neq n} G_m$, and one has

$$a_k = \frac{\tau_k - \pi_k - \lambda}{\pi_k} + \frac{\tau_k - \lambda}{\pi_k}\left(\sqrt[+]{1 - \gamma_k^2/4(\tau_k - \lambda)^2} - 1\right).$$

Fix any compact subset Λ of $\mathbb{C} \setminus \bigcup_{m \neq n} G_m$, then there exists an $K_\Lambda \geqslant 1$ such that for all $|k| \geqslant K_\Lambda$

$$\left|\frac{\tau_k - \lambda}{\pi_k}\left(\sqrt[+]{1 - \gamma_k^2/4(\tau_k - \lambda)^2} - 1\right)\right| \leqslant \left|\frac{\tau_k - \lambda}{\pi_k}\right| \left|\frac{\gamma_k^2}{4(\tau_k - \lambda)^2}\right| \leqslant \left|\frac{\gamma_k}{k}\right| \left|\frac{\gamma_k}{\tau_k - \lambda}\right|,$$

uniformly on Λ. In particular,

$$a_k + a_{-k} = \frac{(\tau_k - \pi_k) - (\tau_{-k} - \pi_{-k})}{\pi_k} + \frac{\ell_k^{p/2}}{k^2} = \frac{\ell_k^p}{k},$$

and

$$a_k a_{-m} = \frac{O(1)}{k^2} + \frac{\ell_k^{p/2}}{k^3} + \frac{\ell_k^{p/4}}{k^4}.$$

This implies $a_k + a_{-k} = \ell_k^1$ and $a_k a_{-k} = \ell_k^1$, hence the infinite product

$$\prod_{k \neq 0} \frac{w_k(\lambda)}{\pi_k} = \prod_{k \geqslant 1} (1 + a_k + a_{-k} + a_k a_{-k})$$

converges absolutely and locally uniformly to a function which is analytic in λ on $\mathbb{C} \setminus \bigcup_{m \neq n} G_m$ and analytic in both variables on $(\mathbb{C} \setminus \bigcup_{m \neq n} \overline{U_m}) \times V_\varphi$ – see [23, Theorem A.4]. ∎

An immediate consequence is a result about the analyticity of quotients of infinite products of the form discussed in Appendix D and f_n.

Corollary 10.6 *For any $\varphi \in \hat{\mathcal{W}}^p$, $\sigma - \sigma^0 \in \ell^p$, and $n \in \mathbb{Z}$,*

$$\prod_{m \neq n} \frac{\sigma_m - \lambda}{w_m(\lambda)}$$

is analytic on $(\mathbb{C} \setminus \bigcup_{m \neq n} \overline{U_m}) \times \ell^p \times V_\varphi$. ⋊

We are now in a position to extend the canonical root. By the definitions of w_n and f_n, and Lemma 8.1 (iv),

$$\Delta^2(\lambda) - 4 = -4 w_n^2(\lambda) f_n^2(\lambda).$$

This allows us to define the canonical root for any $\varphi \in \hat{\mathcal{W}}^p$ by

$$\sqrt[c]{\Delta^2(\lambda) - 4} \coloneqq 2\mathrm{i} w_n(\lambda) f_n(\lambda) = 2\mathrm{i} \prod_{m \in \mathbb{Z}} \frac{w_m(\lambda)}{\pi_m}. \tag{2.13}$$

Lemma 10.7 *For any $\varphi \in \hat{\mathcal{W}}^p$ the canonical root is analytic in λ on $\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} G_m$ and analytic in both variables on $(\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} \overline{U_m}) \times V_\varphi$.* ⋊

In particular, if $\gamma_n = 0$, then $\sqrt[c]{\Delta^2(\lambda) - 4} = 2\mathrm{i}(\tau_n - \lambda) f_n(\lambda)$ is analytic on U_n, while for $\gamma_n \neq 0$ the canonical root extends continuously to each side of G_n and admits opposite signs on opposite sides of G_n,

$$\sqrt[c]{\Delta^2(\lambda) - 4}\,\Big|_{G_n^+} = -\sqrt[c]{\Delta^2(\lambda) - 4}\,\Big|_{G_n^-}. \tag{2.14}$$

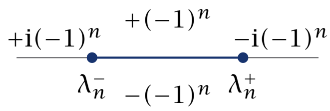

Figure 2.4.: Signs of $\sqrt[c]{\Delta^2(\lambda) - 4}$ around G_n.

Our next result is an adaption of [23, Lemma 12.10] providing an asymptotic estimate of functions of the form considered in Corollary 10.6 which will be used in numerous places of this work.

Lemma 10.8 *For any $\varphi \in \hat{\mathcal{W}}^p$, $1 < p < \infty$, and $\sigma - \sigma^0 \in \ell^p$ with $\sigma - \tau \in \ell^q$, $q \geqslant 1$,*

$$\sup_{\lambda \in U_n} \left| \prod_{m\neq n} \frac{\sigma_m - \lambda}{w_m(\lambda)} - 1 \right| = \ell_n^q + \ell_n^{p/2} + \ell_n^{1+},$$

locally uniformly on $\ell^p \times V_\varphi$. As a consequence,

$$\frac{\sin\lambda}{\lambda - n\pi}\left(\frac{1}{\pi_n}\prod_{m\neq n}\frac{w_m(\lambda)}{\pi_m}\right)^{-1} = 1 + \ell_n^q + \ell_n^{p/2} + \ell_n^{1+}, \qquad \lambda \in U_n$$

locally uniformly on V_φ. ⋊

Proof. By (2.7) there exists $c > 0$ so that for all $m \neq n$ and $\lambda \in U_n$

$$\frac{|\gamma_m|^2}{4|\tau_m - \lambda|^2} \leqslant \frac{c^2}{4}\frac{|\gamma_m|^2}{|m-n|^2}.$$

Since $\gamma_m = \ell_m^p$, we can choose $N \geqslant 0$ so that for all $m \in \mathbb{Z}$ and all $|n| \geqslant N$

$$\frac{c^2}{4}\frac{|\gamma_m|^2}{|m-n|^2} \leqslant \begin{cases} c^2\|\gamma\|_{p/2}^2/|n|^2 \leqslant 1/2, & |m-n| \geqslant |n|/2, \\ c^2\|R_{|n|/2}\gamma\|_{p/2}^2 \leqslant 1/2, & 1 \leqslant |m-n| < |n|/2. \end{cases}$$

Furthermore, for any $m \neq n$ and $\lambda \in U_n$

$$\frac{\sigma_m - \lambda}{w_m(\lambda)} = \frac{\sigma_m - \lambda}{\tau_m - \lambda}\left(1 - \frac{\gamma_m^2}{4(\tau_m - \lambda)^2}\right)^{-1/2},$$

hence the product $\prod_{m\neq n}\frac{\sigma_m - \lambda}{w_m(\lambda)}$ can be written in the form

$$\left(\prod_{m\neq n}\frac{\sigma_m - \lambda}{\tau_m - \lambda}\right)\left(\prod_{m\neq n}\left(1 - \frac{\gamma_m^2}{4(\tau_m - \lambda)^2}\right)^{-1/2}\right).$$

To estimate the first term one can argue as in the proof of Lemma D.6 to obtain

$$\prod_{m\neq n}\frac{\sigma_m - \lambda}{\tau_m - \lambda} - 1\,\Big|_{U_n} = \ell_n^q + \ell_n^{1+}.$$

To estimate the second term we use the estimate $|(1-x)^{-1/2}-1| \leqslant |x|$ for $|x| \leqslant 1/2$ which gives for $|n| \geqslant N$

$$\left|\left(1-\frac{\gamma_m^2}{4(\tau_m-\lambda)^2}\right)^{-1/2}-1\right| \leqslant \left|\frac{\gamma_m^2}{4(\tau_m-\lambda)^2}\right| \leqslant \frac{c^2}{4}\frac{|\gamma_m|^2}{|m-n|^2} \leqslant \frac{1}{2}.$$

Let $S_n = \sum_{m\neq n} \frac{|\gamma_m|^2}{|m-n|^2}$. If $p \geqslant 2$, we can apply Young's inequality (B.1) to find $S_n = \ell_n^{p/2}$, while for $1<p<2$ the basic inequality $(|a|+|b|)^{p/2} \leqslant |a|^{p/2}+|b|^{p/2}$ gives $S_n = \ell_n^{p/2}$. Hence by Lemma D.1 we get

$$\sup_{\lambda\in U_n}\left|\prod_{m\neq n}\left(1-\frac{\gamma_m^2}{4(\tau_m-\lambda)^2}\right)^{-1/2}-1\right| = \ell_n^{p/2}.$$

Going through the arguments of the proof one verifies that the estimate of each factor holds locally uniformly on $\ell^p \times V_\varphi$. ∎

Remark 10.9. This estimate is optimal with respect to p if $p>2$ in the following sense: Suppose $\sigma_m = \tau_m$ for all m then in view of Remark D.7 of Appendix D

$$\sup_{\lambda\in U_n}\left|\prod_{m\neq n}\frac{\sigma_m-\lambda}{w_m(\lambda)}-1-\sum_{m\neq n}\frac{\gamma_m^2}{(m-n)^2}\right| = \ell_n^{p/4}+\ell_n^{1+},$$

while the term $\sum_{m\neq n}\frac{\gamma_m^2}{(m-n)^2}$ is $\ell_n^{p/2}$ but not better. ⊸

The next results improves the estimate of $\lambda_n^\bullet - \tau_n$ given in Lemma 8.6 so that it is also valid for $|n|$ small.

Lemma 10.10 *Locally uniformly on $\hat{\mathcal{W}}^p$, $1<p<\infty$,*

$$\lambda_n^\bullet = \tau_n + \gamma_n^2\ell_n^p.$$

In more detail, for any potential in $\hat{\mathcal{W}}^p$ there exists $C>0$ and a neighborhood V in $\hat{\mathcal{W}}^p$ such that for any φ in V there is a sequence $(a_n)_{n\in\mathbb{Z}}$ with $\sum_{n\in\mathbb{Z}}|a_n|^p<\infty$ and $|\lambda_n^\bullet-\tau_n| \leqslant |\gamma_n^2 a_n|$ for any $n\in\mathbb{Z}$. ⋊

Proof. By Lemma 8.6 for any potential in $\hat{\mathcal{W}}^p$ there exists $C>0$, $N\geqslant 0$, and a neighborhood V such that $\lambda_n^\bullet - \tau_n = \gamma_n^2 a_n$ for $|n|\geqslant N$ and

$$\sum_{|n|\geqslant N}|a_n|^p \leqslant C, \qquad \varphi\in V.$$

By the same arguments as in [23, Lemma 12.11], after possibly shrinking V,

$$\lambda_n^\bullet - \tau_n = O(\gamma_n^2)$$

uniformly on V. This completes the proof. ∎

We conclude this section by investigating the quotient of $\Delta^\bullet$ and the canonical root - c.f. [59, Lemma 2.2]. In view of (2.6) and (2.13) this quotient admits the product representation

$$\frac{\Delta^\bullet(\lambda,\psi)}{\sqrt[c]{\Delta^2(\lambda,\psi)-4}} = -\mathrm{i}\prod_{m\in\mathbb{Z}}\frac{\lambda_m^\bullet(\psi)-\lambda}{w_m(\lambda,\psi)}. \tag{2.15}$$

By Corollary 10.6, for any $\varphi\in\hat{\mathcal{W}}^p$, the quotient is analytic in λ on $\mathbb{C}\setminus\bigcup_{n\in\mathbb{Z}}G_n$ and analytic in both variables on $(\mathbb{C}\setminus\bigcup_{n\in\mathbb{Z}}\overline{U_n})\times V_\varphi$. We call a path in the complex plane *admissible* for $\varphi\in FL_c^p$ if, except possibly at its endpoints, it does not intersect any gap $G_m(\varphi)$.

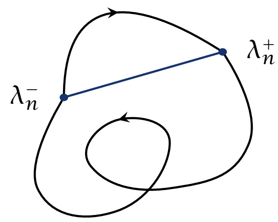

Figure 2.5.: Two admissible paths from λ_n^- to λ_n^+.

Lemma 10.11 *For each $\varphi \in \hat{\mathcal{W}}^p$, $1 < p < \infty$,*

(i) *the function $\frac{\Delta^\bullet(\lambda,\psi)}{\sqrt[c]{\Delta^2(\lambda,\psi)-4}}$ extends analytically to $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$,*

(ii) *for any admissible path from λ_n^- to λ_n^+ in U_n,*

$$\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda,\psi)}{\sqrt[c]{\Delta^2(\lambda,\psi)-4}}\,\mathrm{d}\lambda = 0.$$

In particular, for any closed circuit Γ_n in U_n around G_n,

$$\int_{\Gamma_n} \frac{\Delta^\bullet(\lambda,\psi)}{\sqrt[c]{\Delta^2(\lambda,\psi)-4}}\,\mathrm{d}\lambda = 0. \quad \rtimes$$

Proof. (i) Clearly, $\frac{\Delta^\bullet(\lambda,\psi)}{\sqrt[c]{\Delta^2(\lambda,\psi)-4}}$ is analytic on $\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} G_m$. Furthermore, if $\gamma_n = 0$, then $\lambda_n^\bullet = \tau_n$ by Lemma 10.10. Hence the nth term in the product representation equals one,

$$\frac{\Delta^\bullet(\lambda,\psi)}{\sqrt[c]{\Delta^2(\lambda,\psi)-4}} = -\mathrm{i} \prod_{m\neq n} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)},$$

and by Corollary 10.6 the quotient extends analytically to all of U_n.

(ii) We first consider the case of φ being of real type. Clearly, the functional $\psi \mapsto \int_{\partial U_n} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda$ is analytic on V_φ. Further, for any $\psi \in V_\varphi$ of real type, one has $(-1)^n\Delta(\lambda) \geqslant 2$ on G_n. After deforming the contour of integration to G_n, we therefore get according to the definition of the canonical root – see Figure 2.4 –

$$\int_{\partial U_n} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = 2\int_{\lambda_n^-}^{\lambda_n^+} \frac{(-1)^{n+1}\Delta^\bullet(\lambda)}{\sqrt[+]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = -2\cosh^{-1}\frac{(-1)^n\Delta(\lambda)}{2}\bigg|_{\lambda_n^-}^{\lambda_n^+} = 0.$$

Thus $\int_{\partial U_n} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda$ vanishes on $V_\varphi \cap FL_r^p$ and hence on all of V_φ by Lemma E.2. It follows from the Cauchy Theorem that $\int_{\Gamma_n} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = 0$ for any closed circuit Γ_n in U_n around G_n. In fact, since $\hat{\mathcal{W}}^p = \bigcup_{\varphi\in FL_r^p} V_\varphi$, the claim holds true on all of $\hat{\mathcal{W}}^p$.

Now fix $\varphi \in \hat{\mathcal{W}}^p$ arbitrary. The identity $\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} = 0$ clearly holds in the case $\lambda_n^+ = \lambda_n^-$. If $\lambda_n^+ \neq \lambda_n^-$, then the canonical root admits opposite signs on the opposite sides $G_n^\pm$ of G_n

$$\sqrt[c]{\Delta^2(\lambda)-4}\Big|_{G_n^-} = -\sqrt[c]{\Delta^2(\lambda)-4}\Big|_{G_n^+}.$$

Defining the contour Γ_n by going from λ_n^- to λ_n^+ along the right hand side G_n^- of G_n and then going back from λ_n^+ to λ_n^- along the left hand side G_n^+ of G_n gives

$$0 = \int_{\Gamma_n} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} = 2\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}\big|_{G_n^-}}\,\mathrm{d}\lambda = -2\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}\big|_{G_n^+}}\,\mathrm{d}\lambda.$$

By contour deformation it then follows that $\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} = 0$ along any admissible path in U_n. ∎

11. Actions

For any $\varphi \in \hat{\mathcal{W}}^p$ we have a set of isolating neighborhoods $(U_m)_{m\in\mathbb{Z}}$ which work for a whole neighborhood V_φ of φ. Moreover, by the locally uniform asymptotic behavior of the periodic eigenvalues, we can choose a common set of counterclockwise oriented circuits Γ_n, $n \in \mathbb{Z}$, which are contained in U_n, and a common set of open neighborhoods U_n' of Γ_n so that Γ_n circles around G_n and $\overline{U_n'} \subset U_n \setminus G_n$ for any potential in V_φ. The action variables of φ are now defined by

$$I_n \coloneqq \frac{1}{\pi}\int_{\Gamma_n} \frac{\lambda \Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}, \qquad n \in \mathbb{Z}. \tag{2.16}$$

In view of Lemma 10.11, the integrand is analytic on $U_n' \times V_\varphi$ for each $n \in \mathbb{Z}$, hence by the Cauchy Theorem the definition of the actions is independent of the chosen circuit Γ_n and the particular choice of the isolating neighborhoods.

Lemma 11.1 *Each action I_n is analytic on $\hat{\mathcal{W}}^p$, $1 < p < \infty$, with gradient*

$$\partial I_n = -\frac{1}{\pi}\int_{\Gamma_n} \frac{\partial\Delta(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda. \tag{2.17}$$

On FL^p_r each I_n is real, nonnegative, and vanishes if and only if γ_n vanishes. ⋊

Proof. The analyticity of I_n follows immediately from Lemma 10.11. According to [23, Theorem 13.1] the identity (2.17) holds on $\hat{\mathcal{W}}^2$ and hence extends to $\hat{\mathcal{W}}^p$, $p > 2$, since both hand sides are analytic in φ. If $\gamma_n = 0$, then the integrand of the action I_n is analytic on U_n by Lemma 10.11, so $I_n = 0$. Suppose φ is of real type, and $\gamma_n \neq 0$, then using Lemma 10.11 we may write the actions in the form

$$I_n = \frac{1}{\pi}\int_{\Gamma_n} \frac{(\lambda-\lambda_n^\bullet)\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = \frac{2}{\pi}\int_{\lambda_n^-}^{\lambda_n^+} \frac{(-1)^{n+1}(\lambda-\lambda_n^\bullet)\Delta^\bullet(\lambda)}{\sqrt[+]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda,$$

where in the second step we deformed the contour of integration to the interval G_n and took into account the signs of the canonical root – see Figure 2.4. Since $(-1)^{n+1}(\lambda-\lambda_n^\bullet)\Delta^\bullet(\lambda) \geqslant 0$ on G_n, the integrand is strictly positive on the interior of the gap G_n. Therefore, I_n is real and strictly positive. ∎

We introduce the set

$$Z_n = \{\varphi \in \hat{\mathcal{W}}^p : \gamma_n^2(\varphi) = 0\},$$

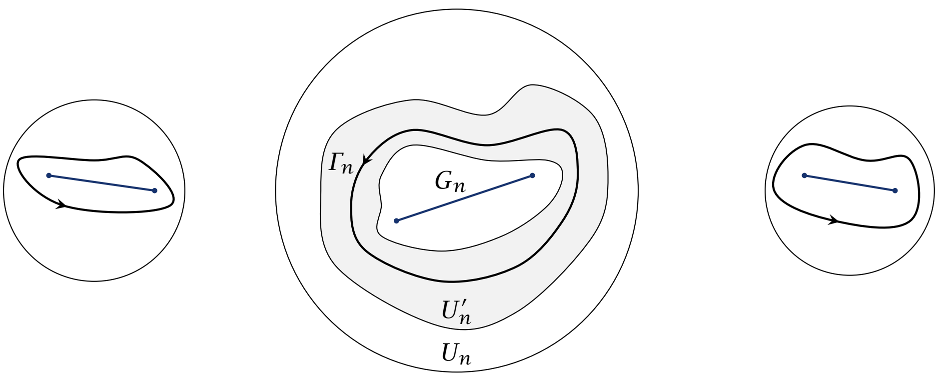

Figure 2.6.: The circuits Γ_n.

of potentials with a collapsed nth gap. By Lemma 10.2 the function γ_n^2 is analytic on $\hat{\mathcal{W}}^p$, hence Z_n is an analytic subvariety of $\hat{\mathcal{W}}^p$. The next result shows that there exists a common neighborhood of FL_r^p so that each quotient I_n/γ_n^2, $n \in \mathbb{Z}$, defined a-priori only on $\hat{\mathcal{W}}^p \setminus Z_n$, admits a real analytic extension to that neighborhood.

Theorem 11.2 *For any $1 < p < \infty$ there exists a complex neighborhood $\mathcal{W}^p \subset \hat{\mathcal{W}}^p$ of FL_r^p, on which each quotient I_n/γ_n^2 is analytic and locally uniformly of the form*

$$\frac{4I_n}{\gamma_n^2} = 1 + \ell_n^{p/2} + \ell_n^{1+}.$$

The real part of I_n/γ_n^2 is positive and locally uniformly bounded away from zero so that

$$\xi_n \coloneqq \sqrt[+]{4I_n/\gamma_n^2}$$

is a well defined real analytic nonvanishing function on $\mathcal{W}^p$ satisfying $\xi_n = 1 + \ell_n^{p/2} + \ell_n^{1+}$ locally uniformly. At the zero potential $\xi_n = 1$ for all $n \in \mathbb{Z}$. ∎

Proof. In view of (2.15) we may write for any $n \in \mathbb{Z}$

$$\frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} = -\mathrm{i}\frac{\lambda_n^\bullet - \lambda}{w_n(\lambda)}\chi_n(\lambda), \qquad \chi_n(\lambda) \coloneqq \prod_{m \neq n} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}, \tag{2.18}$$

where in view of Corollary 10.6 the function χ_n is analytic on $(\mathbb{C} \setminus \bigcup_{m\neq n} \overline{U_n}) \times V_\varphi$. For the action I_n one hence finds

$$I_n = \frac{\mathrm{i}}{\pi}\int_{\Gamma_n} \frac{(\lambda_n^\bullet - \lambda)^2}{w_n(\lambda)}\chi_n(\lambda)\,\mathrm{d}\lambda.$$

On $V_\varphi \setminus Z_n$ we may shrink the contour of integration to the interval G_n and use the parametrization $\lambda_t = \tau_n + t\gamma_n/2$ of G_n together with $w_n|_{G_n^\pm}(\lambda_t) = \mp\mathrm{i}\frac{\gamma_n}{2}\sqrt[+]{1-t^2}$ from (2.12) to obtain

$$I_n = \frac{2}{\pi}\int_{-1}^{1} \frac{(\lambda_n^\bullet - \tau_n - t\gamma_n/2)^2}{\sqrt[+]{1-t^2}}\chi_n(\lambda_t)\,\mathrm{d}t.$$

Consequently,

$$\frac{4I_n}{\gamma_n^2} = \frac{2}{\pi}\int_{-1}^{1} \frac{(t_n - t)^2}{\sqrt[+]{1-t^2}}\chi_n(\lambda_t)\,\mathrm{d}t, \qquad t_n = \frac{\lambda_n^\bullet - \tau_n}{\gamma_n/2}. \tag{2.19}$$

Since $t_n = \gamma_n \ell_n^p = O(\gamma_n)$ by Lemma 10.10, we conclude that for $\gamma_n \to 0$,

$$\frac{4I_n}{\gamma_n^2} \to \frac{2}{\pi}\int_{-1}^{1} \frac{t^2}{\sqrt[+]{1-t^2}}\chi_n(\tau_n)\,\mathrm{d}t = \chi_n(\tau_n).$$

Therefore, I_n/γ_n^2 extends continuously to all of V_φ. Since τ_n is analytic on V_φ and χ_n is analytic on $U_n \times V_\varphi$, also $\chi_n(\tau_n)$ is analytic on V_φ. We conclude with [23, Theorem A.6] that I_n/γ_n^2 extends analytically to all of V_φ.

Since $\lambda_m^\bullet - \tau_m = \gamma_m^2\ell_m^p = \ell_m^{p/2}$ locally uniformly on $\hat{\mathcal{W}}^p$, we conclude with Lemma 10.8 that

$$\sup_{\lambda \in G_n} |\chi_n(\lambda) - 1| = \ell_n^{p/2} + \ell_n^{1+}$$

locally uniformly on $\hat{\mathcal{W}}^p$. Therefore, since $t_n = \ell_n^{p/2}$,

$$\frac{4I_n}{\gamma_n^2} = \frac{2}{\pi}\int_{-1}^{1} \frac{(t_n - t)^2}{\sqrt[+]{1-t^2}}\,\mathrm{d}t + \ell_n^{p/2} + \ell_n^{1+} = 1 + \ell_n^{p/2} + \ell_n^{1+},$$

locally uniformly on $\hat{\mathcal{W}}^p$. Finally, on FL_r^p we have locally uniformly as $|n| \to \infty$

$$0 < \frac{4I_n}{\gamma_n^2} \to 1,$$

which, together with the fact that $\xi_n^2 = \chi_n(\tau_n)$ for $\gamma_n = 0$, gives the remaining claims. ∎

12. Psi-Functions

The goal for this section is to show that there exists a common neighborhood $\mathcal{W}^p \subset \hat{\mathcal{W}}^p$ of FL_r^p so that for any potential $\varphi \in \mathcal{W}^p$ there exists a family of entire functions of the form

$$\psi_n(\lambda) = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}, \qquad \sigma_k^n = k\pi + \ell_k^p, \tag{2.20}$$

satisfying for all $m, n \in \mathbb{Z}$

$$\frac{1}{2\pi} \int_{\Gamma_m} \frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} \, \mathrm{d}\lambda = \delta_{nm}.$$

We essentially follow the construction laid out in [23, Section 14], adapting each step to the situation where $\varphi \in \hat{\mathcal{W}}^p$ carefully analyzing the decay properties of the involved quantities.

As in the introduction of the previous section, given $\varphi \in \hat{\mathcal{W}}^p$, we choose circuits Γ_m, $m \in \mathbb{Z}$, and open neighborhoods U'_m of Γ_m such that Γ_m circles around G_m and $\overline{U'_m} \subset U_m \setminus G_m$ for any potential in V_φ. Then w_m is analytic on $U'_m \times V_\varphi$ for any $m \in \mathbb{Z}$.

Theorem 12.1 *For any $1 < p < \infty$ there exists an open neighborhood $\mathcal{W}^p \subset \hat{\mathcal{W}}^p$ of FL_r^p such that each potential in $\mathcal{W}^p$ admits a family of entire functions ψ_n, $n \in \mathbb{Z}$, of the form (2.20) such that*

$$\frac{1}{2\pi} \int_{\Gamma_m} \frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} \, \mathrm{d}\lambda = \delta_{nm}, \qquad m, n \in \mathbb{Z}. \tag{2.21}$$

The possibly complex roots σ_k^n, $k \neq n$, of ψ_n depend real analytically on φ such that

$$\sigma_k^n = \tau_k + \gamma_k^2 \ell_k^p$$

uniformly in n and locally uniformly on $\mathcal{W}^p$. If φ is of real type, then $\lambda_k^- \leqslant \sigma_k^n \leqslant \lambda_k^+$ for all $k \neq n$. In particular, $\sigma_k^n = k\pi$ for $\varphi = 0$. ⋊

We formulate this goal as a functional statement and apply the implicit function theorem. Given $\varphi \in \hat{\mathcal{W}}^p$ and $n \in \mathbb{Z}$, we are looking for a solution $s = (s_k)_{k \neq n}$ in $\ell^p_{\hat{n},\mathbb{C}} := \ell^p(\mathbb{Z} \setminus \{n\}, \mathbb{C})$ to the equation

$$F^n(\varphi, s) = 0$$

where $F^n = (F^n_m)_{m \neq n}$ is defined by

$$F^n_m(\varphi, s) = A^n_m(\varphi) f_n(s) := (n - m) \int_{\Gamma_m} \frac{f_n(s, \lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} \, \mathrm{d}\lambda \tag{2.22}$$

with $f_n(\lambda) \equiv f_n(s, \lambda)$ given by

$$f_n(s, \lambda) = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\sigma_k - \lambda}{\pi_k}, \qquad \sigma_k := \pi k + s_k. \tag{2.23}$$

We note the following simple observation whose prove is identical to the one of Lemma [23, Lemma 14.2].

Lemma 12.2 *Let $m \neq n$. Suppose φ is in FL_r^p, $1 < p < \infty$, and f is real analytic in a neighborhood of the interval G_m. If $A_m^n f = 0$, then f has a root in G_m.*

A crucial ingredient into the construction is the following estimate, which follows from the mean-value theorem – c.f. Lemma [23, Lemma 14.3].

Lemma 12.3 *If φ is in $\hat{\mathcal{W}}^p$, $1 < p < \infty$, and f is analytic in a neighborhood of G_m containing Γ_m, then*

$$\frac{1}{2\pi}\left|\int_{\Gamma_m} \frac{f(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda\right| \leqslant \max_{\lambda\in G_m}|f(\lambda)|.$$

Moreover, if G_m is a real interval and f is real analytic, then for some $\mu \in G_m$

$$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_m} \frac{f(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda = -f(\mu). \qquad \rtimes$$

In a first step we establish the analyticity of the maps F^n, $n \in \mathbb{Z}$.

Lemma 12.4 *For each $1 < p < \infty$ and $n \in \mathbb{Z}$, formula (2.22) defines a real analytic map*

$$F^n\colon \hat{\mathcal{W}}^p \times \ell^p_{\hat{n},\mathbb{C}} \to \ell^p_{\hat{n},\mathbb{C}}, \qquad F^n(\varphi,s) = (F_m^n(\varphi,s))_{m\neq n}.$$

The maps F^n, $n \in \mathbb{Z}$, are locally uniformly bounded on $\hat{\mathcal{W}}^p \times \ell^p_{\hat{n},\mathbb{C}}$ and uniformly in n. More to the point, $\sum_{m\neq n}|F_m^n(\varphi,s)|^p < C$ where $C > 0$ can be chosen locally uniformly on $\hat{\mathcal{W}}^p \times \ell^p_{\hat{n},\mathbb{C}}$ and uniformly in n. $\rtimes$

To simplify notation we write

$$\frac{f_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} = \frac{\sigma_m - \lambda}{w_m(\lambda)}\zeta_m^n(\lambda), \tag{2.24}$$

where by the product representations (2.23) and (2.13)

$$\zeta_m^n = \frac{\mathrm{i}}{w_n(\lambda)}\prod_{k\neq m,n}\frac{\sigma_k - \lambda}{w_k(\lambda)}, \qquad \sigma_k = k\pi + s_k \in U_k.$$

Sometimes it is more convenient to write $\zeta_m^n(\lambda) = \frac{1}{\mathrm{i}}\frac{1}{\sigma_n - \lambda}\zeta_m(\lambda)$ with

$$\zeta_m(\lambda) = -\prod_{k\neq m}\frac{\sigma_k - \lambda}{w_k(\lambda)}, \tag{2.25}$$

where we have set $\sigma_n = n\pi + s_n$ so that $\sigma_n \in U_n$. Since $s_k = \ell_k^p$ by assumption, Lemma 10.8 yields

$$\zeta_m(\lambda)\big|_{U_m} = -1 + \ell_m^p$$

locally uniformly on $V_\varphi \times \ell^p_{\mathbb{C}}$. Since for $m \neq n$

$$\left.\frac{\sigma_n - \lambda}{n - m}\right|_{U_m} = \pi + O\left(\frac{1}{n-m}\right),$$

locally uniformly on $\ell^p_{\mathbb{C}}$ and uniformly in n. We conclude that

$$(n-m)\zeta_m^n\big|_{U_m} = \frac{\mathrm{i}}{\pi} + O\left(\frac{1}{n-m}\right) + \ell_m^p, \tag{2.26}$$

locally uniformly on $V_\varphi \times \ell^p_{\mathbb{C}}$ and uniformly in n.

Proof. Fix an arbitrary $n \in \mathbb{Z}$ then in view of (2.22) and (2.24)

$$F_m^n = (n-m) \int_{\Gamma_m} \frac{\sigma_m - \lambda}{w_m(\lambda)} \zeta_m^n(\lambda) \, \mathrm{d}\lambda. \tag{2.27}$$

After possibly shrinking V_φ we can choose an open neighborhood U_m' of Γ_m such that $\overline{U_m'} \subset U_m \setminus G_m$ for any potential in V_φ. In particular, w_m is analytic on $U_m' \times V_\varphi$ by Lemma 10.3 and does not vanish there. By Corollary 10.6, ζ_m^n is analytic on $U_m' \times V_\varphi \times \ell_{\mathbb{C},\hat{n}}^p$. Consequently, the integrand is analytic on $U_m' \times V_\varphi \times \ell_{\mathbb{C},\hat{n}}^p$ for any $m \neq n$, hence F_m^n is analytic on $\hat{\mathcal{W}}^p \times \ell_{\mathbb{C},\hat{n}}^p$. Moreover, by Lemma 12.3 and estimate (2.26),

$$F_m^n = O\left(\max_{\lambda \in G_m} |\sigma_m - \lambda| \right)$$

locally uniformly on $\hat{\mathcal{W}}^p \times \ell_{\mathbb{C},\hat{n}}^p$ and uniformly in n. Since $\sigma_m = \tau_m + \ell_m^p$ and $\max_{\lambda \in G_m} |\tau_m - \lambda| = |\gamma_m/2|$, we conclude

$$\sup_{n \neq m} |F_m^n| = \ell_m^p.$$

The map $F^n \colon \hat{\mathcal{W}}^p \times \ell_{\mathbb{C},\hat{n}}^p \to \ell_{\mathbb{C},\hat{n}}^p$ is thus locally bounded and hence analytic on $\hat{\mathcal{W}}^p \times \ell_{\mathbb{C},\hat{n}}^p$. The uniform boundedness with respect to n follows immediately.

It remains to show that F_m^n is real valued on $\mathcal{F}L_r^p \times \ell_{\hat{n}}^p$. Note that if φ is of real type, then $w_m(\lambda)$ is real valued for $\lambda \in \mathbb{R} \setminus G_m$. Consequently, ζ_m is real analytic on U_m, hence

$$F_m^n = \frac{(n-m)}{\mathrm{i}} \int_{\Gamma_m} \frac{1}{w_m(\lambda)} \frac{\sigma_m - \lambda}{\sigma_n - \lambda} \zeta_m(\lambda) \, \mathrm{d}\lambda$$

is real valued in view of Lemma 12.3. ∎

We proceed by investigating the Jacobian of F^n. For our purposes it suffices to restrict ourselves to the open domain $\Omega^p \subset \mathcal{F}L_r^p \times \ell_{\mathbb{R}}^p$ of points (φ, s) with

$$\sigma_m = m\pi + s_m \in U_m, \quad m \in \mathbb{Z}.$$

By analyticity the Jacobian of F^n is a bounded linear operator Q^n on $\ell_{\hat{n}}^p$ which may be represented by an infinite matrix (Q_{mr}^n) with elements

$$Q_{mr}^n = \frac{\partial F_m^n}{\partial s_r}, \qquad m, r \neq n.$$

Lemma 12.5 *On Ω^p, $1 < p < \infty$, the matrix elements Q_{mr}^n of Q^n are real and satisfy for $m, r \neq n$*

$$0 \neq Q_{mm}^n = 2 + O\left(\frac{1}{m-n}\right) + \ell_m^p, \qquad Q_{mr}^n = \frac{\ell_m^p}{m-r}, \quad m \neq r,$$

locally uniformly on Ω^p and uniformly in n. ⋊

Proof. By Lemma 12.4 the elements Q_{mr}^n are real on Ω^p. Moreover, for all $m, r \neq n$ with $m \neq r$

$$\frac{\partial \zeta_m^n}{\partial s_r} = \frac{\zeta_m^n}{\sigma_r - \lambda}$$

so that

$$Q_{mr}^n = (n-m) \int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sigma_r - \lambda} \frac{\zeta_m^n(\lambda)}{w_m(\lambda)} \, \mathrm{d}\lambda.$$

By Lemma 12.3 and (2.26), $\sup_{n\neq m}|Q^n_{mr}|$ is of the order of

$$\max_{\lambda\in G_m}\left|\frac{\sigma_m-\lambda}{\sigma_r-\lambda}\right| = \frac{\ell^p_m}{|m-r|}$$

locally uniformly on Ω^p. For $m = r$ the same arguments lead to

$$Q^n_{mm} = (n-m)\int_{\Gamma_m}\frac{\zeta^n_m(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda.$$

In this case, Lemma 12.3 and (2.26) leads with some $\mu\in G_m$ to

$$Q^n_{mm} = -2\pi\mathrm{i}(n-m)\zeta^n_m(\mu) = 2 + O\left(\frac{1}{m-n}\right) + \ell^p_m,$$

locally uniformly on Ω^p and uniformly in n. The roots of ζ^n_m are precisely the σ_k with $k\neq m,n$. Thus ζ^n_m does not vanish on G_n since $\sigma_k\in U_k$ by assumption. So $Q^n_{mm}\neq 0$ for all $m\neq n$. ∎

Lemma 12.6 *At any point in Ω^p, $1<p<\infty$, the Jacobian Q^n of F^n is of the form*

$$Q^n = D^n + K^n\colon \ell^p_{\hat n}\to\ell^p_{\hat n}$$

where D^n is a linear isomorphism in diagonal form and K^n is a compact operator. ⋊

Proof. Let D^n be the diagonal of Q^n then by Lemma 12.5

$$0\neq Q^n_{mm}\to 2,\qquad |m|\to\infty,$$

so D^n has a bounded inverse on $\ell^p_{\hat n}$. Moreover, $K^n = Q^n - D^n$ is a bounded linear operator on $\ell^p_{\hat n}$ with vanishing diagonal elements and

$$K^n_{mr} = Q^n_{mr} = \frac{\ell^p_m}{m-r},\qquad m\neq r.$$

Since

$$\sum_m\left(\sum_r|K^n_{mr}|^{p'}\right)^{p/p'} = \sum_m\ell^1_m\left(\sum_{r\neq m}\frac{1}{|r-m|^{p'}}\right)^{p/p'} < \infty,$$

the operator K^n can be uniformly approximated by finite rank operators and is hence compact. ∎

Lemma 12.7 *At each point in Ω^p, $1<p<\infty$, each Jacobian Q^n is one-to-one on $\ell^p_{\hat n}$.* ⋊

Proof. Suppose that $Q^n h = 0$ for some $h\in\ell^p_{\hat n}$. By Lemma D.4, f_n, defined in (2.23) is an analytic function on $\mathbb{C}\times\ell^p_{\hat n,\mathbb{C}}$. In particular, $\phi_n(\lambda) \coloneqq \partial_\varepsilon\big|_{\varepsilon=0} f_n(\lambda, s+\varepsilon h)$ is an entire function and for each $m\neq n$

$$0 = \sum_{r\neq n} Q^n_{mr}h_r = (n-m)\int_{\Gamma_m}\frac{\phi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda.$$

Since $\phi_n(\lambda)$ is real analytic, it follows from Lemma 12.2 that $\phi_n(\lambda)$ has a root ρ_m in each G_m with $m\neq n$. The circle $C_\nu = \{|\lambda| = \nu\pi+\pi/2\}$ is contained in $\mathbb{C}\setminus\bigcup_{k\in\mathbb{Z}}\overline{U_k}$ provided $\nu\in\mathbb{N}$ is sufficiently large. Thus $\sigma_r-\lambda\neq 0$ for any $r\in\mathbb{Z}$ and $\lambda\in C_\nu$ such that

$$\phi_n(\lambda) = \sum_{r\neq n}(\partial_{s_r}f_n(\lambda))h_r = f_n(\lambda)\sum_{r\neq n}\frac{h_r}{\sigma_r-\lambda}.$$

Using the product representation of the sine we obtain

$$f_n(\lambda)\frac{\sigma_n-\lambda}{2\sin\lambda}=\prod_{m\in\mathbb{Z}}\frac{\sigma_m-\lambda}{m\pi-\lambda},$$

hence for $\lambda\in C_\nu$ the quotient of $\phi(\lambda)=\phi_n(\lambda)(\sigma_n-\lambda)/2$ and $\sin\lambda$ can be written as

$$\frac{\phi(\lambda)}{\sin\lambda}=\phi_n(\lambda)\frac{\sigma_n-\lambda}{2\sin\lambda}=\sum_{r\neq n}\frac{h_r}{\sigma_r-\lambda}\prod_{m\in\mathbb{Z}}\frac{\sigma_m-\lambda}{m\pi-\lambda}.$$

The infinite product on the right hand side is of the order $1+o(1)$ for $\lambda\in C_\nu$ as $\nu\to\infty$ by Lemma D.5. Moreover, $\sup_{\lambda\in C_\nu}\frac{1}{|\sigma_r-\lambda|}=O\left(\frac{1}{r-\nu}\right)$, hence by Hölder's inequality

$$\sup_{\lambda\in C_\nu}\left|\sum_{r\neq n}\frac{h_r}{\sigma_r-\lambda}\right|\leqslant C_p\left(\sum_{|r|>\nu/2}|h_r|^p\right)^{1/p}+\|h\|_p\left(\sum_{|r|\leqslant\nu/2}\frac{1}{|\sigma_r-\nu\pi|^{p'}}\right)^{1/p'}\to 0,$$

as $\nu\to\infty$. Consequently, $\frac{\phi(\lambda)}{\sin\lambda}=o(1)$ on the circles C_ν as $\nu\to\infty$. Since ϕ vanishes at $\rho_m=m\pi+\ell^p_m$ for $m\neq n$ as well as at σ_n it follows from the Interpolation Lemma E.1 that ϕ and hence ϕ_n vanishes identically, which implies $h=0$. ∎

The preceding two lemmas together with the Fredholm alternative prove the following.

Corollary 12.8 *At each point in Ω^p, $1<p<\infty$, each Jacobian Q^n, $n\in\mathbb{Z}$, is a linear isomorphism of $\ell^p_{\hat n}$.* ⋊

We are now in a position to invoke the implicit function theorem to construct the solutions s^n of the equation $F^n(\varphi,s^n)=0$.

Proposition 12.9 *For $1<p<\infty$ and any $n\in\mathbb{Z}$ there exists a unique real analytic map*

$$s^n\colon FL^p_r\to\ell^p_{\hat n}$$

with graph in $\Omega^p_{\hat n}$ such that $F^n(\varphi,s^n(\varphi))\equiv 0$ everywhere. Moreover, for any $\varphi\in FL^p_r$ we have $\sigma^n_m(\varphi)\in G_m$ for all $m\neq n$. ⋊

Proof. Suppose $(\varphi,s)\in\Omega^p_{\hat n}$ is a solution of $F(\varphi,s)=0$. It follows from Lemma 12.2 that f_n has a root ρ_m in each G_m, $m\neq n$. Since the G_m are pairwise disjoint and $\lambda^\pm_m=m\pi+\ell^p_m$ it follows that these are the only roots of f_n hence $\sigma_m=\rho_m\in G_m$.

By Lemma 12.4, Corollary 12.8, and the implicit function theorem, any solution (φ^0,s^0) of the equation $F(\varphi,s)=0$ can be locally extended so that s is a real analytic function of φ in a neighborhood of φ^0. We claim that by the continuation method this map can be extended to any $\varphi\in FL^p_r$. To this end, suppose (φ^k,s^k) is a sequence in Ω^p with $F(\varphi^k,s^k)=0$ so that $\varphi^k\to\varphi$ in FL^p_r. Clearly, the endpoints of $G_m(\varphi^k)$ converge to the end points of $G_m(\varphi)$ by Lemma 8.2. Moreover, since for any $m\in\mathbb{Z}$ the interval $G_m(\varphi^k)$, $k\geqslant 1$, is compact, after possibly passing to a subsequence, also $\sigma_m(\varphi^k)$ is convergent as $k\to\infty$. Let $\sigma_m(\varphi)$ denote the limit, then $\sigma_m(\varphi)\in G_m(\varphi)$ and $F^n_m(\varphi,s(\varphi))=0$ with $s(\varphi)=(\sigma_m(\varphi)-m\pi)_{m\neq n}$. Hence $(\varphi,s(\varphi))\in\Omega^p$ and we can apply the implicit function theorem at $(\varphi,s(\varphi))$. This shows that the continuation method applies. Since FL^p_r is simply connected, any particular solution (φ^0,s^0) of $F^n(\varphi,s)=0$ thus extends uniquely and globally to a real analytic map $s^n\colon FL^p_r\to\ell^p_{\hat n}$ with graph in $\Omega^p_{\hat n}$ satisfying $F^n(\varphi,s^n(\varphi))=0$ everywhere.

At $\varphi=0$ one solution is given by $s(0)=0$ as one can verify straightfowardly using Cauchy's formula. This solution is also unique, because $G_m(0)=\{m\pi\}$ for any $m\in\mathbb{Z}$. Hence there is exactly one such analytic map. ∎

Lemma 12.10 *All real analytic maps $s^n\colon FL_r^p \to \ell_{\hat{n}}^p$, $1 < p < \infty$, of Proposition 12.9 extend to a complex neighborhood $\mathcal{W}^p \subset \hat{\mathcal{W}}^p$ of FL_r^p which is independent of n so that for any potential $\varphi \in FL_r^p$ and any $n \in \mathbb{Z}$, the restriction of the solution s^n to $\mathcal{W}^p \cap V_\varphi$ satisfies $\sigma_m^n \in U_m$ for any $m \neq n$.* ⋊

Proof. In a first step we show that for each $(\varphi, s) \in \Omega^p$ the inverse of the Jacobian Q^n at $(\varphi, (s_k)_{k\neq n})$ is uniformly bounded with respect to $n \in \mathbb{Z}$. First recall that for $m, r \neq n$

$$Q_{mr}^n = \frac{1}{\mathrm{i}\pi}\int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sigma_r - \lambda}\frac{n\pi - m\pi}{\sigma_n - \lambda}\frac{\zeta_m(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda,$$

where we have chosen s_n so that $\sigma_n = n\pi + s_n \in U_n$. Then we have uniformly for λ in U_m

$$\frac{n\pi - m\pi}{\sigma_n - \lambda} = 1 + \frac{(n\pi - \sigma_n) + (\lambda - m\pi)}{\sigma_n - \lambda} = 1 + O\left(\frac{1}{n-m}\right). \tag{2.28}$$

In particular,

$$\lim_{n\to\infty} Q_{mr}^n = Q_{mr}^* = \frac{1}{\mathrm{i}\pi}\int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sigma_r - \lambda}\frac{\zeta_m(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda.$$

Note that the right hand side is well defined for all $m, r \in \mathbb{Z}$ including the case where $m, r = n$. By the same arguments as in the proof of Lemma 12.5 one concludes

$$0 \neq Q_{mm}^* = 2 + \ell_m^p, \qquad Q_{mr}^* = \frac{\ell_m^p}{m-r}, \quad m \neq r, \tag{2.29}$$

hence these elements define a bounded linear operator on $\ell_{\mathbb{R}}^p$ which we denote by Q^*. We now also view Q^n as an operator on $\ell_{\mathbb{R}}^p$ by setting $Q_{nn}^n = 2$ and $Q_{mn}^n = Q_{nm}^n = 0$ for $m \neq n$. We claim $Q^n \to Q^*$ in the ℓ^p-operator norm.

For any $h \in \ell_{\mathbb{R}}^p$

$$\|(D^* - D^n)h\|_p^p = |(Q_{nn}^* - 2)h_n|^p + \sum_{m\neq n}\left|\frac{1}{\mathrm{i}\pi}\int_{\Gamma_m}\left(\frac{n\pi - m\pi}{\sigma_n - \lambda} - 1\right)\frac{\zeta_m(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda\, h_m\right|^p.$$

First note that

$$|(Q_{nn}^* - 2)h_n|^p = \ell_n^1\|h\|_p^p.$$

Furthermore, $\frac{n\pi - m\pi}{\sigma_n - \lambda} - 1 = \frac{n\pi - \sigma_n}{\sigma_n - \lambda} + \frac{\lambda - m\pi}{\sigma_n - \lambda}$ and one has by Lemma 12.3

$$\int_{\Gamma_m}\frac{n\pi - \sigma_n}{\sigma_n - \lambda}\frac{\zeta_m(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda = O(s_n), \qquad \int_{\Gamma_m}\frac{\lambda - m\pi}{\sigma_n - \lambda}\frac{\zeta_m(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda = O\left(\frac{|\gamma_m| + |\tau_m - m\pi|}{|n-m|}\right),$$

locally uniformly on Ω^p, hence

$$\begin{aligned}
&\sum_{m\neq n}\left|\frac{1}{\mathrm{i}\pi}\int_{\Gamma_m}\left(\frac{n\pi - m\pi}{\sigma_n - \lambda} - 1\right)\frac{\zeta_m(\lambda)}{w_m(\lambda)}\,\mathrm{d}\lambda h_m\right|^p \\
&\quad = \|h\|_p^p O\left(\ell_n^1 + \sup_{m\neq n}\frac{|\gamma_m|^p + |\tau_m - m\pi|^p}{|m-n|^p}\right) \\
&\quad = \|h\|_p^p O\left(\ell_n^1 + \frac{1}{|n|^p} + \sum_{|m-n|\geqslant|n|/2}(|\gamma_m|^p + |\tau_m - m\pi|^p)\right).
\end{aligned}$$

So we conclude $\|D^* - D^n\|_p \to 0$ as $n \to \infty$. In the sequel, we show by similar arguments that also $\|K^* - K^n\|_p \to 0$ as $n \to \infty$. For any $h \in \ell_{\mathbb{R}}^p$

$$\|(K^* - K^n)h\|_p^p = \left|\sum_{r\neq n}(Q_{nr}^* - Q_{nr}^n)h_r\right|^p + \sum_{m\neq n}\left|\sum_{r\neq m}(Q_{mr}^* - Q_{mr}^n)h_r\right|^p.$$

For the first summand, using $Q^n_{nr} = 0$ and (2.29), we obtain with Hölder's inequality

$$\left| \sum_{r \neq n} (Q^*_{nr} - Q^n_{nr}) h_r \right|^p = \ell^1_n \left| \sum_{r\neq n} \frac{h_r}{n-r} \right|^p \leqslant \ell^1_n \|h\|^p_p.$$

The second summand is decomposed as

$$\sum_{m\neq n} \left| \sum_{r\neq m} (Q^*_{mr} - Q^n_{mr}) h_r \right|^p \leqslant I + II,$$

where

$$I = \sum_{m\neq n} \left| \sum_{r\neq m,n} \frac{1}{\pi} \int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sigma_r - \lambda} \left(\frac{n\pi - m\pi}{\sigma_n - \lambda} - 1 \right) \frac{\zeta_m(\lambda)}{w_m(\lambda)} \,\mathrm{d}\lambda \, h_r \right|^p,$$
$$II = \sum_{m\neq n} \left| \frac{1}{\mathrm{i}\pi} \int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sigma_n - \lambda} \frac{\zeta_m(\lambda)}{w_m(\lambda)} \,\mathrm{d}\lambda \, h_n \right|^p.$$

For the second term we obtain

$$\begin{aligned} II &= \|h\|^p_p \, O\left(\sup_{m\neq n} \frac{|\gamma_m|^p + |\tau_m - \sigma_m|^p}{|m-n|^p} \right) \\ &= \|h\|^p_p \, O\left(\frac{1}{|n|^p} + \sum_{|m-n|\geqslant |n|/2} (|\gamma_m|^p + |\tau_m - \sigma_m|^p) \right), \end{aligned}$$

while for the first term

$$\begin{aligned} &\left| \frac{1}{\pi} \int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sigma_r - \lambda} \left(\frac{n\pi - m\pi}{\sigma_n - \lambda} - 1 \right) \frac{\zeta_m(\lambda)}{w_m(\lambda)} \,\mathrm{d}\lambda \right| \\ &\leqslant \sup_{\lambda \in G_m} \left| \frac{\sigma_m - \lambda}{\sigma_r - \lambda} \right| \frac{|\sigma_n - n\pi| + |\lambda - m\pi|}{|\sigma_n - \lambda|} = \frac{\ell^p_m}{|r-m|} \frac{\ell^p_n + \ell^p_m}{|n-m|} = \frac{\ell^p_m}{|n-m|} \frac{1}{|r-m|}, \end{aligned}$$

and hence

$$I = \sum_{m\neq n} \frac{\ell^1_m}{|n-m|^p} \left| \sum_{r\neq m,n} \frac{h_r}{r-m} \right|^p \leqslant O\left(\frac{1}{|n|^p} + \sum_{|m|\geqslant |n|/2} \ell^1_m \right) \|h\|^p_p.$$

Thus we conclude $\|K^* - K^n\|_p \to 0$ as $n \to \infty$. By going through the arguments of the proof one checks that the convergence of $Q^n = D^n + K^n$ to $Q^* = D^* + K^*$ is locally uniform on Ω^p. For each $n \in \mathbb{Z}$, Q^n is a continuous map on Ω^p with values in the space of linear operators on $\ell^p_{\mathbb{R}}$. By the locally uniform convergence Q^* is continuous on Ω^p as well. Moreover, by the same arguments as in the proof of Lemma 12.7, it follows that Q^* is boundedly invertible at each point in Ω^p. Since the set

$$\Pi(\varphi) = \prod_{m\in\mathbb{Z}} \big(G_m(\varphi) - m\pi \big)$$

is compact in $\ell^p_{\mathbb{R}}$, the operator $Q^*(\varphi, s)$ is indeed uniformly boundedly invertible for $s = (s_m)_{m\in\mathbb{Z}} \in \Pi(\varphi)$. By continuity also $Q^n(\varphi, s)$ is uniformly boundedly invertible for all sufficiently large n and all $s \in \Pi(\varphi)$ and hence for all n by Corollary 12.8. This concludes the first step of the proof.

By Lemma 12.4 the map F^n is locally uniformly bounded on $\hat{\mathcal{W}}^p \times \ell^p_{\hat{n},\mathbb{C}}$ and uniformly with respect to n. It follows from Cauchy's estimate that for every φ there exists a neighborhood V of $\varphi \times \Pi(\varphi)$ such that

$$\|Q^n(\psi, t) - Q^n(\varphi, s)\| \leqslant O(\|\psi - \varphi\|), \quad (\psi, t) \in V,$$

where the implicit constant can be chosen uniformly on V and uniformly in n. After possibly shrinking V, we may apply the standard estimate

$$\|(Q+\delta Q)^{-1}\| \leqslant 2\|Q^{-1}\| \text{ for all } \delta Q \text{ with } \|Q^{-1}\delta Q\| \leqslant \frac{1}{2}$$

to $Q = Q^n(\varphi, s)$ to obtain a bound of the inverse of the Jacobian Q^n which is uniform with respect to n and locally uniform on an open simply connected neighborhood of Ω^p which is contained in

$$\bigcup_{\varphi\in FL^p_r} V_\varphi \times \{s \in \ell^p_{\mathbb{C}} : \sigma_m \in U_m(\varphi) \text{ for all } m \in \mathbb{Z}\},$$

where $(U_m)_{m\in\mathbb{Z}}$ is a sequence of isolating neighborhoods of φ. The solutions s^n obtained in Proposition 12.9 can thus be extended using the implicit function theorem to an n independent open simply connected neighborhood $\mathcal{W}^p \subset \hat{\mathcal{W}}^p$ of FL^p_r such that for every $\psi \in V_\varphi \cap W$ with $\varphi \in FL^p_r$

$$\sigma^n_m(\psi) = m\pi + s^n_m(\psi) \in U_m(\varphi), \qquad m \neq n. \qquad \blacksquare$$

We have thus shown that each potential in $\mathcal{W}^p$ admits a family of entire functions $\psi_n = f_n(s^n)$ whose roots σ^n_k, $k \neq n$, depend real analytically on φ and are contained in U_k. Moreover, the functions ψ_n satisfy the integral equations (2.21) for $m \neq n$. Next we check this condition also for $m = n$.

Lemma 12.11 *For any $\varphi \in \mathcal{W}^p$, $1 < p < \infty$, and any $n \in \mathbb{Z}$*

$$\int_{\Gamma_n} \frac{f_n(s^n,\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = 2\pi. \qquad \rtimes$$

Proof. Fix $\varphi \in \mathcal{W}^p$ and $n \in \mathbb{Z}$. The integrals over the cycles Γ_m, $m \neq n$, vanish by the construction of the solution s^n. Hence,

$$\int_{\Gamma_n} \frac{f_n(s^n,\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = \int_{C_\nu} \frac{f_n(s^n,\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda$$

for the circles $C_\nu = \{|\lambda| = \nu\pi + \pi/2\}$ provided $\nu > |n|$ is sufficiently large. Recall that $\sigma^n_m = m\pi + s^n_m(\varphi)$ for $m \neq n$, hence by setting $\sigma^n_n = \tau_n$ we obtain from (2.23)

$$\frac{f_n(s^n,\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} = \frac{\mathrm{i}}{\sigma_n-\lambda}\prod_{m\in\mathbb{Z}}\frac{\sigma_m-\lambda}{w_m(\lambda)}.$$

Since $\sigma^n_m = m\pi + \ell^p_m$ by the construction of the solution s^n, Lemma D.5 shows

$$\prod_{m\in\mathbb{Z}}\frac{\sigma^n_m-\lambda}{\pi_m}\Big|_{C_\nu} = (1+o(1))\sin\lambda\big|_{C_\nu}, \quad \text{as} \quad \nu\to\infty.$$

Similarly, one concludes $\prod_{m\in\mathbb{Z}}\frac{w_m(\lambda)}{\pi_m}\big|_{C_\nu} = (1+o(1))\sin\lambda\big|_{C_\nu}$ as $\nu\to\infty$, implying that

$$\prod_{m\in\mathbb{Z}}\frac{\sigma^n_m-\lambda}{w_m(\lambda)}\Big|_{C_\nu} = 1+o(1), \qquad \nu\to\infty.$$

Consequently,

$$\lim_{\nu\to\infty}\int_{C_\nu}\frac{f_n(s^n,\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = \lim_{\nu\to\infty}\int_{C_\nu}\frac{\mathrm{i}}{\sigma_n-\lambda}\,\mathrm{d}\lambda = 2\pi. \qquad \blacksquare$$

To complete the proof of Theorem 12.1 it remains to establish the asymptotics of the roots σ^n_m of ψ_n.

Lemma 12.12 *On the complex neighborhood $\mathcal{W}^p \subset \hat{\mathcal{W}}^p$, $1 < p < \infty$, of FL^p_r of Lemma 12.10*

$$\sigma_m^n = \tau_m + \gamma_m^2 \ell_m^p[n]$$

uniformly in n and locally uniformly on $\mathcal{W}^p$. In more detail, there exists a sequence α_m^n so that $\sigma_m^n = \tau_m + \gamma_m^2 \alpha_m^n$ where

$$\sum_{m\in\mathbb{Z}} |\alpha_m^n|^p \leqslant C,$$

and $C > 0$ can be chosen uniformly in n and locally uniformly on $\mathcal{W}^p$. ⋊

Proof. Let $\varphi \in \mathcal{W}^p$ and let $(U_m)_{m\in\mathbb{Z}}$ be a sequence of isolating neighborhoods which work for a neighborhood $V_\varphi \subset \mathcal{W}^p$ of φ. Given a solution s^n of $F^n(\varphi, s^n)$ we obtain from rewriting (2.27)

$$0 = \int_{\Gamma_m} \frac{\sigma_m^n - \lambda}{w_m(\lambda)} \chi_m^n(\lambda)\,\mathrm{d}\lambda, \qquad m \neq n. \tag{2.30}$$

Here χ_m^n is the analytic function on U_m, $m \neq n$, given by

$$\chi_m^n(\lambda) = \frac{\pi}{\mathrm{i}} \frac{n-m}{\sigma_n^n - \lambda} \zeta_m(\lambda), \tag{2.31}$$

with ζ_m defined as in (2.25) and we set $\sigma_n^n = \tau_n$. It follows from (2.26) that

$$\chi_m^n(\lambda)\big|_{U_m} = \mathrm{i} + O\left(\frac{1}{n-m}\right) + \ell_m^p[n] = \mathrm{i} + \ell_m^p[n], \qquad m \neq n, \tag{2.32}$$

meaning that $|\chi_m^n - \mathrm{i}|_{U_m} = \alpha_m^n$ where $\sum_{m\neq n} |\alpha_m^n|^p \leqslant C$ and the constant C can be chosen uniformly in n. Expanding χ_m^n at $\lambda = \tau_m$ up to first order gives

$$\chi_m^n(\lambda) = \chi_m^n(\tau_m) + (\lambda - \tau_m) b_m^n(\lambda).$$

The sequence of circuits Γ_m can be chosen so that $\inf_{m\in\mathbb{Z}} \operatorname{dist}(\Gamma_m, \partial U_m) > 0$. Consequently, by Cauchy's estimate

$$\sup_{\lambda\in\Gamma_m} |b_m^n(\lambda)| = \ell_m^p[n]$$

locally uniformly on V_φ where the implicit constant is uniform in n. Arguing as in the proof of Lemma 10.3 shows that

$$\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_m} \frac{\sigma_m^n - \lambda}{w_m(\lambda)}\,\mathrm{d}\lambda = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_m} \frac{\sigma_m^n - \tau_m}{w_m(\lambda)}\,\mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_m} \frac{\tau_m - \lambda}{w_m(\lambda)}\,\mathrm{d}\lambda = \tau_m - \sigma_m^n.$$

Inserting the expansion of χ_m^n and the above expression into (2.30) gives

$$\chi_m^n(\tau_m)(\sigma_m^n - \tau_m) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_m} \frac{(\sigma_m^n - \lambda)(\lambda - \tau_m)}{w_m(\lambda)} b_m^n(\lambda)\,\mathrm{d}\lambda. \tag{2.33}$$

In view of (2.31) there exists $c > 0$ so that

$$|\chi_m^n(\tau_m)| \geqslant c|\zeta_m(\tau_m)|, \qquad n \neq m.$$

Since $\zeta_m(\tau_m) = -1 + \ell_m^p$ and ζ_m does not vanish on G_m we conclude $\inf_{n\in\mathbb{Z}, m\neq n} |\chi_m^n(\tau_m)| > 0$. As a consequence, for all $n \in \mathbb{Z}$ and $m \neq n$ by (2.33)

$$\begin{aligned} |\sigma_m^n - \tau_m| &\leqslant \max_{\lambda\in G_m} |\sigma_m^n - \lambda| |\gamma_m| \ell_m^p[n] \\ &\leqslant (|\sigma_m^n - \tau_m| + |\gamma_m|/2) |\gamma_m| \ell_m^p[n]. \end{aligned} \tag{2.34}$$

Since $\sigma_m^n \in U_m$ we have $\sigma_m^n - \tau_m = O(1)$ uniformly in $m \neq n$. Substituting this estimate into the right hand side of (2.34) gives $\sigma_m^n - \tau_m = \gamma_m \ell_m^p[n]$ locally uniformly on $\mathcal{W}^p$ and uniformly in n. Reinserting this estimate into (2.34) then yields $\sigma_m^n - \tau_m = \gamma_m^2 \ell_m^p[n]$ locally uniformly on $\mathcal{W}^p$ and uniformly in n. ∎

13. Angles

Let $\mathcal{W}^p$ denote the common complex neighborhood of FL_r^p on which the functions ψ_n, $n \in \mathbb{Z}$, exist according to Theorem 12.1. Following [23, Section 15], one can define the angular coordinate θ_n for potentials φ in $\mathcal{W}^p \setminus Z_n$ by the formula

$$\theta_n(\varphi) = \eta_n(\varphi) + \sum_{m \neq n} \beta_m^n(\varphi)$$

where

$$\eta_n(\varphi) = \int_{\lambda_n^-}^{\mu_n} \frac{\psi_n(\lambda)}{\sqrt[*]{\Delta^2(\lambda) - 4}}\,\mathrm{d}\lambda \mod 2\pi$$

and for $m \neq n$ and actually all $\varphi \in \mathcal{W}^p$

$$\beta_m^n(\varphi) = \int_{\lambda_m^-}^{\mu_m} \frac{\psi_n(\lambda)}{\sqrt[*]{\Delta^2(\lambda) - 4}}\,\mathrm{d}\lambda.$$

The integrals are taken along any admissible path from λ_m^- to μ_m inside the isolating neighborhood U_m, that is the paths contain besides their starting point λ_m^- and possibly their endpoint μ_m no point of G_m. For $\mu_m \neq \lambda_m^\pm$ the sign of the square root is chosen such that

$$\sqrt[*]{\Delta^2(\mu_m) - 4} = \delta(\mu_m)$$

while in case $\mu_m = \lambda_m^\pm$ the sign does not matter as in view of (2.21) for $m = n$ the integral is only affected by a multiple of 2π and for $m \neq n$ the integral is zero.

By the same arguments as in [23, Section 15], one proves the following.

Theorem 13.1 *Suppose $1 < p < \infty$.*

(i) *For any $m \neq n$ the functions β_m^n are real analytic functions satisfying*

$$\beta_m^n = O\left(\frac{|\gamma_m| + |\mu_m - \tau_m|}{|m-n|}\right)$$

locally uniformly on $\mathcal{W}^p$.

(ii) *The functions η_n are real analytic on $\mathcal{W}^p \setminus Z_n$ when taken modulo π.*

(iii) *The series $\beta_n = \sum_{m \neq n} \beta_m^n$ converges locally uniformly to a real analytic function on $\mathcal{W}^p$ such that $\beta_n = o(1)$.*

(iv) *The angle function*

$$\theta_n = \eta_n + \beta_n = \eta_n + \sum_{m \neq n} \beta_m^n$$

defined modulo 2π, is a real valued function on FL_r^p and extends to a real analytic function on $\mathcal{W}^p \setminus Z_n$ when taken modulo π. ⋊

This completes the construction of Liouville coordinates in a complex neighborhood of FL_r^p for any $1 < p < \infty$. These coordinates are *canonical* in the following sense.

Corollary 13.2 *For any $1 < p < \infty$ one has*

$$\{I_n, I_m\} = 0 \text{ on } FL_r^p,$$

$$\{\theta_n, \theta_m\} = 0 \text{ on } FL_r^p \setminus (Z_n \cup Z_m),$$

$$\{\theta_n, I_m\} = \delta_{nm} \text{ on } FL_r^p \setminus Z_n. \quad ⋊$$

Proof. According to [23, Section 18] these identities hold for $p = 2$ and hence extend to $p > 2$ by continuity. ∎

Chapter 3

Birkhoff coordinates

14. Overview

The main result of the previous chapter is that there exists a complex neighborhood $\mathcal{W}^p$ of FL_r^p, $1 < p < \infty$, on which (1) each action variable I_n, $n \in \mathbb{Z}$, is analytic and satisfies the identity

$$4I_n = \xi_n^2 \gamma_n^2,$$

and (2) each angular coordinate $\theta_n = \eta_n + \beta_n$ is analytic on $\mathcal{W}^p \setminus Z_n$ when considered modulo π where $Z_n = \{\varphi \in \mathcal{W}^p : \gamma_n^2(\varphi) = 0\}$ is an analytic subvariety of $\mathcal{W}^p$.

Consequently, on $\mathcal{W}^p \setminus Z_n$ the rectangular coordinates

$$x_n = \frac{\xi_n \gamma_n}{\sqrt{2}} \cos \theta_n, \qquad y_n = \frac{\xi_n \gamma_n}{\sqrt{2}} \sin \theta_n, \tag{3.1}$$

are well defined. In this section we extend these rectangular coordinates real analytically to all of $\mathcal{W}^p$ and show that they constitute to a map

$$\Omega_p : \varphi \mapsto (x_n(\varphi), y_n(\varphi))_{n \in \mathbb{Z}},$$

which has the following properties.

Theorem 14.1 *For any* $1 < p < \infty$,

(i) $\Omega_p : \mathcal{W}^p \to \ell_c^p \coloneqq \ell_{\mathbb{C}}^p \times \ell_{\mathbb{C}}^p$ *is real analytic,*

(ii) for any $1 < q < p$ *the restriction* $\Omega_p|_{\mathcal{W}_q}$ *coincides with* Ω_q *on* $\mathcal{W}^p \cap \mathcal{W}_q$,

(iii) Ω_p *is canonical,*

(iv) the map $\Omega_p : FL_r^p \to \ell_r^p \coloneqq \ell_{\mathbb{R}}^p \times \ell_{\mathbb{R}}^p$ *is one-to-one, a local diffeomorphism at every point of* FL_r^p, *and the image of this map is open and dense in* ℓ_r^p.

(v) for $1 < p \leqslant 2$ *the map* $\Omega_p : FL_r^p \to \ell_r^p$ *is also onto and hence is a bi-real analytic diffeomorphism.* ⋊

Remark 14.2. (i) For $p = 2$, the map Ω_2 coincides with the map Ω constructed in [23].

(ii) At this point, the surjectivity of the Birkhoff map Ω_p for $2 < p < \infty$ is an open question.

(iii) With some efforts, Theorem 14.1 can be extended to $FL_r^{s,p}$ with $s > 0$ and $1 < p < \infty$.

(iv) A crucial ingredient into the construction of the Birkhoff coordinates is that the eigenvalues of the associated spectral problem asymptotically come in isolated pairs. Due to the asymptotic behavior of the eigenvalues, which involves the Fourier coefficients of the potentials, having at least some decay in the Fourier coefficients seems to be the minimal requirement for this construction to work. ⊸

Method of proof. Using arguments from [23, Section 16], we show that the rectangular coordinates defined in (3.1) extend real analytically to a common neighborhood of FL_r^p, $1 < p < \infty$. The asymptotic behavior of the periodic and the Dirichlet spectra obtained in Chapter 1 then implies that the coordinates are locally uniformly ℓ^p-summable and hence constitute to a real analytic map $\Omega_p : FL_r^p \to \ell_r^p$, $\varphi \mapsto (x_n, y_n)_{n\in\mathbb{Z}}$. The canonical relations as well as the injectivity follow by a density argument from the case $p = 2$. Further, we conclude from the asymptotic behavior of the spectral quantities used to define the Birkhoff map that its differential, at every point, is a Fredholm operator of index zero. As a consequence of the canonical relations, the differential is one-to-one and hence an isomorphism, proving that Ω is a local diffeomorphism everywhere. Finally, the surjectivity in the case $1 < p < 2$ follows by an argument of [36] using the flows generated by the coordinate functions to reduce the surjectivity to a neighborhood of the origin where it clearly holds since Ω_p is a local diffeomorphism.

Related results. The construction of the Birkhoff coordinates for dNLS in the case $p = 2$ is described in great detail in [23] – cf. also [3, 4, 55]. Birkhoff coordinates have been constructed also for other equations such as KdV [33, 31, 36], modified KdV [35], and the Toda chain [26]. In [27] the Birkhoff coordinates of KdV were extended to the Fourier Lebesgue spaces $FL^{-1/2,p}$, $2 < p < \infty$, using the global Birkhoff coordinates of KdV on the Hilbert space H^{-1} which contains all the spaces $FL^{-1/2,p}$.

15. Analyticity

As the starting point for obtaining the analyticity of the Birkhoff coordinates we write (3.1) in the more convenient form

$$x_n = \frac{\xi_n \gamma_n}{\sqrt{8}}(\mathrm{e}^{\mathrm{i}\eta_n + \mathrm{i}\beta_n} + \mathrm{e}^{-\mathrm{i}\eta_n - \mathrm{i}\beta_n}), \qquad y_n = \frac{\xi_n \gamma_n}{\sqrt{8}\mathrm{i}}(\mathrm{e}^{\mathrm{i}\eta_n + \mathrm{i}\beta_n} + \mathrm{e}^{-\mathrm{i}\eta_n - \mathrm{i}\beta_n}).$$

By Theorem 11.2 each function ξ_n is analytic on $\mathcal{W}^p$, while by Theorem 13.1 each function β_n is analytic on $\mathcal{W}^p$. Hence it suffices to consider the functions

$$z_n^+ = \gamma_n \mathrm{e}^{\mathrm{i}\eta_n}, \qquad z_n^- = \gamma_n \mathrm{e}^{-\mathrm{i}\eta_n}.$$

Both γ_n and η_n mod 2π have discontinuities. Nevertheless, using that locally around every point in $\mathcal{W}^p \setminus Z_n$ there exist analytic functions ρ_n^+ and ρ_n^- so that the set equality $\{\rho_n^-, \rho_n^+\} = \{\lambda_n^-, \lambda_n^+\}$ holds, one shows that $z_n^\pm$ is actually analytic on $\mathcal{W}^p \setminus Z_n$ – see [23, Lemma 16.1] for further details. One then proceeds to compute $z_n^\pm$ along a sequence of potentials $\varphi^{(k)} \in \mathcal{W}^p \setminus Z_n$ which converge to some $\varphi \in Z_n$. This limit is different from zero, when φ is in the set

$$F_n = \{\psi \in \mathcal{W}^p : \mu_n \notin G_n\},$$

which is an open subset disjoint from FL_r^p. Proving that $z_n^\pm = O(|\gamma_n| + |\mu_n - \tau_n|)$ holds locally uniformly on $\mathcal{W}^p$, one then shows that $z_n^\pm$ extends continuously to all of $\mathcal{W}^p$. Finally, one verifies using the previously obtained values of $z_n^\pm$ on Z_n that the restriction of $z_n^\pm$ to Z_n is weakly analytic. The analyticity of $z_n^\pm$ on $\mathcal{W}^p$ then follows from [23, Lemma A.6] so that we conclude with the following result – see [23, Section 16] for more details.

Lemma 15.1 *For any $n \in \mathbb{Z}$ the functions $z_n^\pm$ are analytic on $\mathcal{W}^p$, $1 < p < \infty$, and the estimate*

$$z_n^\pm = O(|\gamma_n| + |\mu_n - \tau_n|)$$

holds locally uniformly on $\mathcal{W}^p$ and uniformly in n. ⋊

For any $\varphi \in \mathcal{W}^p$, $1 < p < \infty$, we now define the Birkhoff coordinates by

$$x_n = \frac{\xi_n}{\sqrt{8}}(z_n^+ \mathrm{e}^{\mathrm{i}\beta_n} + z_n^- \mathrm{e}^{-\mathrm{i}\beta_n}), \qquad y_n = \frac{\xi_n}{\sqrt{8}\mathrm{i}}(z_n^+ \mathrm{e}^{\mathrm{i}\beta_n} - z_n^- \mathrm{e}^{-\mathrm{i}\beta_n}), \qquad n \in \mathbb{Z}, \tag{3.2}$$

where each function is analytic on $\mathcal{W}^p$. Since we have locally uniformly on $\mathcal{W}^p$ the estimates $\xi_n = 1 + \ell_n^p$ by Theorem 11.2, $\beta_n = o(1)$ by Theorem 13.1, and $z_n^+ = \ell_n^p$ in view of the previous lemma and Theorems 1.2 and 1.5, we conclude $x_n = \ell_n^p$ and $y_n = \ell_n^p$ locally uniformly on $\mathcal{W}^p$. With [23, Theorem A.4] we thus obtain the following.

Theorem 15.2 *For any $1 < p < \infty$, the map*

$$\Omega_p \colon FL_r^p \to \ell_r^p, \qquad \varphi \mapsto (x_n(\varphi), y_n(\varphi))_{n\in\mathbb{Z}},$$

is real analytic and extends to an analytic map $\mathcal{W}^p \to \ell_c^p$. ∎

A first corollary is that the map Ω_p is canonical in the following sense.

Lemma 15.3 *On all of FL_r^p, $1 < p < \infty$,*

$$\{x_n, x_m\} = 0, \qquad \{x_n, y_m\} = -\delta_{nm}, \qquad \{y_n, y_m\} = 0,$$

for all $n, m \in \mathbb{Z}$. ⋊

Proof. All these brackets are real analytic on FL_r^p, and the identities hold for $p = 2$ according to [23, Theorem 18.8]. Hence the identities also hold for the case $1 < p < 2$ by restriction and for the case $p > 2$ by continuity. ∎

16. Jacobian

Since the Birkhoff map $\Omega_p \colon \mathcal{W}^p \to \ell_c^p$ is analytic, its Jacobian is an analytic map

$$\mathrm{d}\Omega_p \colon \mathcal{W}^p \to L(FL_c^p, \ell_c^p), \qquad \varphi \mapsto \mathrm{d}_\varphi\Omega_p,$$

where for any $h \in FL_c^p$

$$\mathrm{d}_\varphi\Omega_p h = (\langle b_n^+, h\rangle_r, \langle b_n^-, h\rangle_r)_{n\in\mathbb{Z}}.$$

Here, $b_n^+ = \partial_\varphi x_n$ and $b_n^- = \partial_\varphi y_n$ denote the L^2-gradients of the functionals x_n and y_n, respectively. It follows from the analyticity that these gradients are elements of $FL_c^{p'}$ with $1/p + 1/p' = 1$. Recall from (1.2) that

$$e_m^+ = \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}m\pi x} \end{pmatrix}, \qquad e_m^- = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}m\pi x} \\ 0 \end{pmatrix}.$$

At the zero potential one has in view of [23, Theorem 17.2] that $b_n^{\pm} = d_n^{\pm}$ where

$$d_n^+ = -\frac{1}{\sqrt{2}}(e_{-2n}^+ + e_{-2n}^-), \qquad d_n^- = -\frac{1}{\sqrt{2}\mathrm{i}}(e_{-2n}^+ - e_{-2n}^-),$$

hence $\mathrm{d}_0\Omega_p$ coincides with the Fourier transform. We proceed by estimating the deviation of the gradients of the Birkhoff coordinates for real valued finite gap potentials from the case of the zero potential. In a first step consider the gradients of the function $z_n^{\pm} = \gamma_n \mathrm{e}^{\pm\mathrm{i}\eta_n}$.

Lemma 16.1 *At any finite gap potential in FL_r^p, $1 < p < \infty$,*

$$\|\partial z_n^{\pm} + 2e_{-2n}^{\pm}\|_{p'} = \ell_n^p. \quad \rtimes$$

Proof. One obtains by the same arguments as in [23, Lemma 17.1] for potential in $FL_r^p \cap Z_n$ the identity

$$\partial z_n^{\pm} = 2(\partial\tau_n - \partial\mu_n) \pm \Big(\mathrm{i}2\delta(\mu_n)\partial\phi_n + 2\phi_n\Big(\mathrm{i}\partial\delta\big|_{\lambda=\mu_n} + \mathrm{i}\dot\delta(\mu_n)\partial\mu_n\Big)\Big), \tag{3.3}$$

where

$$\phi_n = \frac{\zeta_n(\mu_n)}{\psi_n(\mu_n)}, \qquad \zeta_n(\lambda) = -\prod_{m\neq n}\frac{\sigma_m^n - \lambda}{w_m(\lambda)}.$$

Suppose φ is a finite gap potential of real type, then $F = \{n \in \mathbb{Z} : \gamma_n(\varphi) > 0\}$ is a finite set and $\varphi \in H_r^1$. The gradient $\partial z_n^{\pm}$ is an element of $FL_c^{p'}$ since $z_n^{\pm}$ is real analytic on FL_r^p. By Lemma G.7 from the appendix we have on H_r^1

$$\|\partial\tau_n\|_{p'} = \ell_n^p, \qquad \|\partial\mu_n - \frac{1}{2}(e_{-2n}^+ + e_{-2n}^-)\|_{p'} = \ell_n^p.$$

Furthermore, $\mu_n = n\pi + \ell_n^p$, hence by Lemma 9.2

$$\delta(\mu_n) = \ell_n^p, \qquad \dot\delta(\mu_n) = \ell_n^p.$$

In particular, $\|\dot\delta(\mu_n)\partial\mu_n\|_{p'} = \ell_n^p$. Moreover, by Lemma G.6

$$\|\mathrm{i}\partial\delta\big|_{\lambda=\mu_n} - (-1)^n(e_{-2n}^+ - e_{-2n}^-)\|_{p'} = \ell_n^p.$$

Since $\zeta_n(\mu_n) = 1 + \ell_n^p$ by Lemma 10.8 and $\psi_n(\mu_n) = -2(-1)^n + \ell_n^p$ by Lemma D.8, we conclude that $2\phi_n = (-1)^{n-1} + \ell_n^p$. Further, ϕ_n is real analytic on $\mathcal{W}^p$ and the estimate of $2\phi_n - (-1)^{n-1}$ holds locally uniformly on $\mathcal{W}^p$, hence it follows from Cauchy's estimate that $\|\partial\phi_n\|_{p'} = O(1)$. Therefore

$$\|\delta(\mu_n)\partial\phi_n\|_{p'} = \ell_n^p,$$

as well as

$$\Big\|\phi_n\Big(\mathrm{i}\partial\delta\big|_{\lambda=\mu_n} - (-1)^n(e_{-2n}^+ - e_{-2n}^-) + \mathrm{i}\dot\delta(\mu_n)\partial\mu_n\Big)\Big\|_{p'} = \ell_n^p.$$

Altogether we thus find for each $n \in \mathbb{Z}\setminus F$ in view of (3.3)

$$\partial z_n^{\pm} = -(e_{-2n}^+ + e_{-2n}^-) \pm (-1)^{n-1}(-1)^n(e_{-2n}^+ - e_{-2n}^-) + r_n^{\pm} = -2e_{-2n}^{\pm} + r_n^{\pm},$$

where $\|r_n^{\pm}\|_{p'} = \ell_n^p$. ∎

As a consequence, we obtain the following estimate of the gradients of the Birkhoff coordinates at real valued finite gap potentials.

Lemma 16.2 *At any finite gap potential in FL_r^p, $1 < p < \infty$,*

$$\|b_n^+ - d_n^+\|_{p'} = \ell_n^p, \qquad \|b_n^- - d_n^-\|_{p'} = \ell_n^p. \quad \rtimes$$

Proof. Recall from (3.2) that

$$x_n = \frac{\xi_n}{\sqrt{8}}(z_n^+ e^{i\beta_n} + z_n^- e^{-i\beta_n}), \qquad y_n = \frac{\xi_n}{\sqrt{8}i}(z_n^+ e^{i\beta_n} - z_n^- e^{-i\beta_n}).$$

By Theorem 11.2 we have $\xi_n = 1 + \ell_n^p$. If φ is a finite gap potential, then $\beta_n = O(1/n)$ by Theorem 13.1 and $z_n^\pm = 0$ for $|n|$ sufficiently large, where we used that $|\mu_m - \tau_m| \leqslant |\gamma_m|/2$ for φ of real type. Moreover, by Cauchy's estimate $\|\partial\xi_n\|_{p'}, \|\partial z_n^\pm\|_{p'}, \|\partial\beta_n\|_{p'} = O(1)$. Therefore,

$$\left\|\partial x_n - \frac{1}{\sqrt{8}}(\partial z_n^+ + \partial z_n^-)\right\|_{p'} = \ell_n^p, \qquad \left\|\partial z_n - \frac{1}{\sqrt{8}i}(\partial z_n^+ - \partial z_n^-)\right\|_{p'} = \ell_n^p,$$

and the claim follows with Lemma 16.1. ∎

In the sequel we use the fact that the gradients $b_n^\pm$ are close to the ones of the zero potential to prove that the Jacobian $d_\varphi\Omega_p$ is a compact perturbation of the Fourier transform $d_0\Omega_p$. It turns out to simplify matters, if one introduces the map

$$A_\varphi = (d_0\Omega_p)^{-1} d_\varphi\Omega_p,$$

and shows that A_φ is a compact perturbation of the identity, instead. Note that for any $w = (w_n^+, w_n^-)_{n\in\mathbb{Z}}$ one has

$$(d_0\Omega_p)^{-1} w = \sum_{n\in\mathbb{Z}} (w_{-n}^+ d_n^+ + w_{-n}^- d_n^-),$$

and hence for any $h \in FL_c^p$

$$A_\varphi h = \sum_{n\in\mathbb{Z}} (\langle b_{-n}^+, h\rangle_r d_n^+ + \langle b_{-n}^-, h\rangle_r d_n^-). \tag{3.4}$$

Lemma 16.3 *The operator $d_\varphi\Omega_p$ is an isomorphism if and only if the operator A_φ is an isomorphism. Moreover, A_φ is a compact perturbation of the identity which depends real analytically on $\varphi \in FL_r^p$, $1 < p < \infty$.* $\rtimes$

Proof. In view of (3.4) we have

$$(A_\varphi - \mathrm{Id})h = \sum_{n\in\mathbb{Z}} (\langle b_{-n}^+ - d_{-n}^+, h\rangle_r d_n^+ + \langle b_{-n}^- - d_{-n}^-, h\rangle_r d_n^-).$$

Next we show that $S_\varphi \coloneqq A_\varphi - I \colon FL_c^p \to FL_c^p$ is the limit of finite rank operators and hence compact. To this end, define for any $N \geqslant 0$ the finite rank approximation,

$$S_\varphi^N = \sum_{|n|\leqslant N} (\langle b_{-n}^+ - d_{-n}^+, h\rangle_r d_n^+ + \langle b_{-n}^- - d_{-n}^-, h\rangle_r d_n^-),$$

and note that by Hölder's inequality

$$\begin{aligned}\|(S_\varphi - S_\varphi^N)h\|_p^p &= \sum_{|n|>N} (|\langle b_{-n}^+ - d_{-n}^+, h\rangle_r|^p + |\langle b_{-n}^- - d_{-n}^-, h\rangle_r|^p) \\ &\leqslant \sum_{|n|>N} \left(\|b_{-n}^+ - d_{-n}^+\|_{p'}^p + \|b_{-n}^- - d_{-n}^-\|_{p'}^p\right)\|h\|_p^p.\end{aligned}$$

Suppose φ is a finite gap potential of real type, then Lemma 16.2 shows that $\|S_\varphi - S_\varphi^N\|_{FL_c^p \to FL_c^p} \to 0$ as $N \to \infty$, hence S_φ is compact. Since the map S_φ depends real analytically on φ, it follows that S_φ is compact at every $\varphi \in FL_r^p$ and hence A_φ is a compact perturbation of the identity. ∎

17. Diffeomorphism property

We are now in a position to prove the remaining properties claimed in Theorem 14.1.

Proposition 17.1 *For any $1 < p < \infty$ the map*

$$\Omega_p \colon FL_r^p \to \ell_r^p$$

is a local diffeomorphism. ∎

Proof. By Lemma 16.3 the differential $\mathrm{d}_\varphi \Omega_p$ is an isomorphism if and only if A_φ is an isomorphism. Moreover, A_φ is a Fredholm operator of index zero, and so is its adjoint A_φ^*. Let $K = -\mathrm{i}J$ with $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, then $A_\varphi K b_m^\pm = -d_{-m}^\pm$ in view of the canonical relations of Lemma 15.3. Suppose $0 = A_\varphi^* h$, then

$$0 = \langle A_\varphi^* h, K b_m^\mp \rangle_r = \langle h, A_\varphi K b_m^\mp \rangle_r = -\langle h, d_{-m}^\mp \rangle_r.$$

Since the vectors $d_m^\pm$, $m \in \mathbb{Z}$, form a basis of FL_r^p, it follows that $h = 0$. Consequently, A_φ^* is a linear isomorphism and so is A_φ. ∎

Proposition 17.2 *For any $1 < p < \infty$ the map*

$$\Omega_p \colon FL_r^p \to \ell_r^p$$

is one-to-one. ∎

Proof. It has been shown in [23, Theorem 19.3] that $\Omega_p \colon FL_r^2 \to \ell_r^2$ is one-to-one, hence we only have to deal with the case $p > 2$. Suppose Ω_p is not one-to-one, then there exist $\varphi_1 \neq \varphi_2 \in FL_r^p$ with $z = \Omega_p(\varphi_1) = \Omega_p(\varphi_2)$. Since Ω_p is a local diffeomorphism, there exist disjoint open neighborhoods U_1, U_2 of φ_1 and φ_2 in FL_c^p, and an open neighborhood V of z in ℓ_c^p so that $\Omega_p \colon U_1 \to V$ and $\Omega_p \colon U_2 \to V$ are both diffeomorphisms. Moreover, $U_1 \cap L_r^2$ and $U_2 \cap L_r^2$ are open in L_r^2 and $V \cap \ell_r^2$ is open in ℓ_r^2 so that $\Omega_p|_{L_r^2} \colon U_1 \cap L_r^2 \to V \cap \ell_r^2$ and $\Omega_p|_{L_r^2} \colon U_2 \cap L_r^2 \to V \cap \ell_r^2$ are diffeomorphisms. Since $\Omega_p|_{L_r^2}$ is one-to-one it follows that $U_1 = U_2$ which gives a contradiction. ∎

Proposition 17.3 *For any $1 < p \leqslant 2$ the map*

$$\Omega_p \colon FL_r^p \to \ell_r^p$$

is onto. ∎

Proof. The case $p = 2$ is the content of [23, Theorem 19.3]. We proceed by adapting the method presented in [36] to derive the ontoness for $1 < p < 2$ we from the case $p = 2$. Fix any $1 < p < 2$ and any element $z_0 \in \ell_r^p$. Since $\Omega_2 \colon FL_r^2 \to \ell_r^2$ is onto, there exists a preimage $\varphi_0 \in FL_r^2$ so that $\Omega_2(\varphi_0) = z_0$. We assume that

$$\varphi_0 \in FL_r^2 \setminus FL_r^p \tag{3.5}$$

and attribute this to a contradiction. Since $\Omega_p(0) = 0$ and by Proposition 17.1 the differential of Ω_p at 0 is a linear isomorphism, there exists by the inverse function theorem an open neighborhood U of 0 in FL^p_c and an open neighborhood V of 0 in ℓ^p_c so that

$$\Omega_p\big|_U : U \to V$$

is a diffeomorphism. Without loss we can assume that V is a ball of radius 4ε, centered at the origin,

$$V = B_{4\varepsilon}(0) \subset \ell^p_c. \tag{3.6}$$

To obtain the desired contradiction, we construct a finite sequence $\varphi_1, \dots, \varphi_N$ with the property that $\varphi_N \in U$ and $\varphi_n - \varphi_{n-1} \in FL^p_r$ for any $1 \leqslant n \leqslant N$. To this end we consider the action angle variables (I_k, θ_k), $k \in \mathbb{Z}$, constructed in Chapter 2 on $FL^{p'}_r$ where $1/p + 1/p' = 1$ and $1 < p < 2 < p' < \infty$. More to the point, for any $k \in \mathbb{Z}$, the action I_k is defined on $FL^{p'}_r$ and satisfies $I_k = (x_k^2 + y_k^2)/2$ whereas the angle θ_k is defined on $FL^{p'}_r \setminus Z_k$ with values in $\mathbb{R}/2\pi\mathbb{Z}$ and is real analytic when considered modulo π. The L^2-gradient $\partial_\varphi \theta_k$ is a real analytic map on $FL^{p'}_r \setminus Z_k$ with values in FL^p_r. The Hamiltonian vector field $Y_k = -\mathrm{i}J\partial_\varphi \theta_k$, when restricted to FL^2_r, is a real analytic map $Y_k : FL^2_r \setminus Z_k \to FL^p_r$ and defines a dynamical system

$$\dot\varphi = Y_k(\varphi), \qquad \varphi(0) = \varphi_0 \in FL^2_r \setminus D_k. \tag{3.7}$$

We now use the flows of these vector fields to construct the sequence $\varphi_1, \dots, \varphi_N$ recursively as follows. Let $(m_n)_{n\geqslant 1}$ be the sequence of integers $0, -1, 1, -2, 2, \cdots$. The potential φ_0 is given by (3.5). For any $n \geqslant 1$ assume that φ_{n-1} has already been constructed. If $I_{m_n}(\varphi_{n-1}) < \varepsilon^2/2^{2n/p}$, then set $\varphi_n \equiv \varphi_{n-1}$. If $I_{m_n}(\varphi_{n-1}) \geqslant \varepsilon^2/2^{2n/p}$, then $\varphi_{n-1} \in FL^2_r \setminus Z_n$ and hence the vector field Y_n is well defined in a neighborhood of φ_{n-1}. The element φ_n is then chosen to be an element on the solution curve of the vector field Y_{m_n} passing through φ_{n-1} so that $I_{m_n}(\varphi_n) < \varepsilon^2/2^{2n/p}$. The existence of such a potential φ_n follows from Lemma 17.4 (i) below. Moreover, by the commutator relations given in Corollary 13.2,

$$Y_n(I_k) = \{I_k, \theta_n\} = -\delta_{nk},$$

hence the vector field Y_n preserves the value of the action variable I_{m_k} with $k \neq n$. In particular, we have

$$I_{m_k}(\varphi_n) < \varepsilon^2 \frac{1}{2^{2k/p}}, \qquad 1 \leqslant k \leqslant n,$$

and

$$I_{m_k}(\varphi_n) = I_k(\varphi(0)), \qquad k > n.$$

One then obtains

$$\begin{aligned}\|\Omega_2(\varphi_n)\|_p^p &= \sum_{m\in\mathbb{Z}} \big(|x_m(\varphi_n)|^p + |y_m(\varphi_n)|^p\big) \leqslant 2^{p/2} \sum_{m\in\mathbb{Z}} |I_m(\varphi_n)|^{p/2} \\ &\leqslant 2^{p/2}\varepsilon^p \sum_{1\leqslant k\leqslant n} \frac{1}{2^k} + 2^{p/2} \sum_{k>n} |I_{m_k}(\varphi_0)|^{p/2}.\end{aligned}$$

Since $\Omega_2(\varphi_0) \in \ell^p_r$, one has $\sum_{m\in\mathbb{Z}} |I_m(\varphi_0)|^{p/2} < \infty$. Choose $N \geqslant 1$ so that

$$\sum_{|m|>N} |I_m(\varphi_0)|^{p/2} < \varepsilon^p,$$

then $\|\Omega_2(\varphi_N)\|_p^p \leqslant 2^{p/2+1}\varepsilon^p \leqslant (4\varepsilon)^p$. Since $\Omega_p\big|_U : U \to V = B_{4\varepsilon}(0)$ is a diffeomorphism, $\Omega_2\big|_{\mathcal{W}^p} = \Omega_p$, and $\Omega_2(\varphi^{(N)}) \in V$ it then follows that

$$\varphi_N \in U \subset FL^p_r. \tag{3.8}$$

On the other hand, assumption (3.5) implies that $\varphi_0 \in FL^2_r \setminus FL^p_r$, hence $\varphi_N \in FL^2_r \setminus FL^p_r$ by Lemma 17.4 (ii) which contradicts (3.8). Consequently, $\varphi_0 \in FL^p_r$ and the ontoness is proved. ∎

The following lemma, which is used in the proof of the previous result, is an adaption of [36, Lemma 1].

Lemma 17.4 *For any $\varphi_0 \in L^2_r \setminus Z_k$ with $k \in \mathbb{Z}$ the initial value problem $\dot\varphi = Y_k(\varphi)$, $\varphi(0) = \varphi_0$, has a unique solution $t \mapsto \varphi(t)$ in $C^1((-\infty, I_k(\varphi_0)), L^2_r)$. The solution has the following additional properties*

(i) $\lim\limits_{t \nearrow I_k(\varphi_0)} I_k(\varphi(t)) = 0$,

(ii) $t \mapsto \varphi(t) - \varphi_0$ *is in* $C^0((-\infty, I_k(\varphi_0)), FL^p_r)$ *for any* $1 < p < \infty$. ⋊

Proof. The Birkhoff map $\Omega_2 : L^2_r \to \ell^2_r$, $\varphi \mapsto \Omega(\varphi) = (x_n, y_n)_{n\in\mathbb{Z}}$, is a bianalytic diffeomorphism which transforms the Poisson structure on L^2_r into the canonical Poisson structure $\{x_n, y_m\} = -\delta_{nm}$ on ℓ^2_r. Moreover, the angle θ_k is the argument of the complex number $x_k + \mathrm{i}y_k$. Consequently, one has for any $\varphi \in L^2_r \setminus D_k$

$$\mathrm{d}\Omega(Y_k) = \frac{x_k}{x_k^2 + y_k^2}\partial x_k - \frac{y_k}{x_k^2 + y_k^2}\partial y_k. \tag{3.9}$$

The dynamical system corresponding to the vector field (3.9) in ℓ^2_r has a unique solution for any initial datum $(x_n^0, y_n^0)_{n\in\mathbb{Z}}$ which is defined on the time interval $(-\infty, ((x_k^0)^2 + (y_k^0)^2)/2)$. Since $\Omega : L^2_r \to \ell^2_r$ is a diffeomorphism, the dynamical system (3.7) has a unique solution $\varphi(t)$ on $L^2_r \setminus D_k$ defined on $(-\infty, I_k(\varphi_0))$. Moreover, one gets from (3.9) and $2I_n = x_n^2 + y_n^2$ that

$$\lim_{t \nearrow I_k(\varphi_0)} I_k(\varphi(t)) = 0,$$

which proves (i). To prove (ii) we integrate both sides of (3.7) and get for any $t \in (-\infty, I_k(\varphi_0))$

$$\varphi(t) = \varphi_0 + \int_0^t Y_k(\varphi(s))\,\mathrm{d}s.$$

Since $Y_k : L^2_r \setminus D_k \to FL^p_r$ is real analytic for any $1 < p < \infty$ and the solution $\varphi(t)$ is continuous on L^2_r, the integrand is in $C^0((-\infty, I_k(\varphi_0)), FL^p_r)$. In particular, the integral converges in the FL^p_r-norm which proves (ii). ∎

We conclude this section with a brief discussion of the image of the *isospectral set* of $\varphi \in FL^p_r$

$$\mathrm{Iso}(\varphi) = \{\psi \in FL^p_r : \mathrm{spec}_{per}(\psi) = \mathrm{spec}_{per}(\varphi)\},$$

under the transformation $\Omega_p : FL^p_r \to \ell^p_r$ – c.f. [23, Theorem 20.1]. To this end, we introduce the tori

$$\mathrm{Tor}(I) \coloneqq \Big\{(x, y) \in \ell^p_r : (x_n^2 + y_n^2)/2 = I_n \quad \text{for all} \quad n \in \mathbb{Z}\Big\},$$

where $I = I(\varphi)$ is the sequence of actions associated with φ. Note that $\mathrm{Tor}(\varphi)$ is a compact subset of ℓ^p_r for any $\varphi \in FL^p_r$, $1 < p < \infty$.

Lemma 17.5 *(i) For any* $1 < p \leqslant 2$,

$$\Omega_p(\mathrm{Iso}(\varphi)) = \mathrm{Tor}(I(\varphi)).$$

In particular, $\mathrm{Iso}(\varphi)$ *is a compact subset of* FL^p_r.

(ii) For any $p > 2$

$$\Omega_p(\mathrm{Iso}(\varphi)) \subset \mathrm{Tor}(I(\varphi)). \quad \rtimes$$

Proof. Suppose $\varphi \in FL^p_r$. Since any potential $\psi \in \mathrm{Iso}(\varphi)$ has the same periodic spectrum as φ, it also has the same discriminant by Lemma 8.1 and hence the same actions in view of formula (2.16). Consequently, $I(\psi) = I(\varphi)$ so that $\Omega_p(\psi) \in \mathrm{Tor}(I)$ which proves $\Omega_p(\mathrm{Iso}(\varphi)) \subset \mathrm{Tor}(I(\varphi))$.

The converse inclusion has been established in [23, Theorem 20.1] for the case $p = 2$ using the flows generated by the vector fields associated to the actions and the fact that Ω_2 is a global diffeomorphism. In particular, if $\varphi \in FL^p_r$ with $1 < p < 2$, then $\Omega_2^{-1}(\mathrm{Tor}(I(\varphi))) = \mathrm{Iso}(\varphi)$ where φ is viewed as an element of L^2_r. Since $\Omega_2|_{FL^p_r} = \Omega_p$, $I(\varphi) \in \ell^p_r$, and $\Omega_p \colon FL^p_r \to \ell^p_r$ is a global diffeomorphism as well, the claim follows. ∎

Chapter 4

Frequencies, Convexity, and Wellposedness

18. Overview

Let us briefly recall the central notations from the introduction to state the results of this chapter. The NLS Hamiltonian System with Hamiltonian

$$\mathcal{H} = \int_{\mathbb{T}} (\varphi_-' \varphi_+' + \varphi_-^2 \varphi_+^2) \, \mathrm{d}x,$$

is given by

$$\mathrm{i}\partial_t \varphi = J \partial \mathcal{H}, \qquad J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{4.1}$$

When this system is restricted to the invariant subspace L_r^2 of real type states $\varphi = (u, \overline{u})$, one obtains the defocusing NLS equation (dNLS)

$$\mathrm{i}\partial_t u = \partial_{\overline{u}} \mathcal{H}(u, \overline{u}) = -\partial_x^2 u + 2|u|^2 u. \tag{4.2}$$

The Birkhoff map $\Omega \colon \varphi \mapsto (x_n, y_n)_{n\in\mathbb{Z}}$ constructed in the prequel introduces Birkhoff coordinates for the NLS Hamiltonian. More to the point, by [23] the restriction of the Birkhoff map to H_r^m, $m \geqslant 0$ integer, is a canonical real analytic diffeomorphism onto $h_r^m = \ell^{m,2}(\mathbb{Z}, \mathbb{R}) \times \ell^{m,2}(\mathbb{Z}, \mathbb{R})$ and the actions $I = (I_n)_{n\in\mathbb{Z}}$, when restricted to H_r^m, take values on the positive quadrant $\ell_+^{2m,1}$ of $\ell_{\mathbb{R}}^{2m,1} \coloneqq \ell^{2m,1}(\mathbb{Z}, \mathbb{R})$. The NLS Hamiltonian $\mathcal{H}$ is a real analytic map on H_r^1 and, when expressed in Birkhoff coordinates on h_r^1, renders as a function of the actions alone. This function, denoted by H, is a real analytic map $H \colon \ell_+^{2,1} \to \mathbb{R}$ and according to [51] admits an expansion at $I = 0$ of the form

$$H = \sum_{m\in\mathbb{Z}} (2m\pi)^2 I_m + 2H_1^2 - \sum_{m\in\mathbb{Z}} I_m^2 + \cdots, \qquad H_1 = \sum_{m\in\mathbb{Z}} I_m,$$

where the dots stand for higher order terms in I. In particular, at $I = 0$,

$$\partial_{I_n} H\big|_{I=0} = (2n\pi)^2, \qquad \partial_{I_n}\partial_{I_m} H\big|_{I=0} = -2\delta_{nm}, \qquad n, m \in \mathbb{Z}.$$

The NLS frequencies, defined as $\omega_n = \partial_{I_n} H$, thus have an expansion at $I = 0$ of the form

$$\omega_n = (2n\pi)^2 + 4H_1 - 2I_n + \cdots. \tag{4.3}$$

The equations of motion for dNLS in Birkhoff coordinates then take the particularly simple form

$$\dot{x}_n = -\omega_n y_n, \quad \dot{y}_n = \omega_n x_n.$$

To state our first result, we introduce the frequencies $\omega_n^\star$ where we remove from ω_n the part which is constant on level sets of H_1,

$$\omega_n^\star = \omega_n - (2n\pi)^2 - 4H_1.$$

The frequencies $\omega_n^\star$ correspond to the Hamiltonian

$$H^\star = H - 2H_1^2 - \sum_{m\in\mathbb{Z}} (2m\pi)^2 I_m.$$

The expansion (4.3) implies that $\omega_n^\star$ has an expansion at $I = 0$ of the form

$$\omega_n^\star = -2I_n + \cdots .$$

We also introduce the frequency map $\omega^\star = (\omega_n^\star)_{n\in\mathbb{Z}}$. A priori this frequency map is a real analytic map $\omega^\star : \ell_+^{2,1} \to \ell_{\mathbb{R}}^{-2,\infty}$. To extend the domain of analyticity of this map, let $\mathcal{V}^{p/2}$, $p \geqslant 2$, denote the image of $\mathcal{W}^p$ through the map $\mathcal{W}^p \to \ell_{\mathbb{C}}^{p/2}$, $\varphi \mapsto (I_n(\varphi))_{n\in\mathbb{Z}}$. In view of Theorem 14.1, this image is an open subset of $\ell_{\mathbb{C}}^{p/2}$ which contains ℓ_+^1.

Theorem 18.1 *The map $\omega^\star$ is defined on ℓ_+^1, takes values in $\bigcap_{r>1} \ell^r$, and $\omega^\star : \ell_+^1 \to \ell^r$ is real analytic for any $r > 1$. Moreover, for any $p > 2$ the map $\omega^\star$ admits a real analytic extension $\omega^\star : \mathcal{V}^{p/2} \to \ell_{\mathbb{C}}^{p/2}$ with asymptotics*

$$\omega_n^\star + 2I_n = \ell_n^{p/3} + \ell_n^{1+},$$

which hold locally uniformly on $\mathcal{V}^{p/2}$. ⋊

The asymptotics of $\omega_n^\star$ obtained in Theorem 18.1 together with the expansion of $\omega_n^\star$ at $I = 0$ lead to the following result on the frequency map which has possible applications to the analysis of perturbations of the dNLS equation.

Corollary 18.2 *For any $p > 2$,*

(i) $\omega^\star : \mathcal{V}^{p/2} \to \ell^{p/2}$ *is a local diffeomorphism near $I = 0$.*

(ii) For any $I \in \mathcal{V}^{p/2}$, the map $\Lambda_I = \mathrm{d}_I\omega^\star + 2\mathrm{Id}$ is compact.

(iii) $\omega^\star : \mathcal{V}^{p/2} \to \ell^{p/2}$ *is a Fredholm map of index zero everywhere.*

(iv) $\omega^\star : \mathcal{V}^{p/2} \to \ell^{p/2}$ *is a local diffeomorphism generically.* ⋊

Our next result shows that the Hamiltonian $\mathcal{H}^\star = \mathcal{H} - 2\mathcal{H}_1^2$, defined a priori on H_r^1, admits a real analytic extension to FL_r^4. On this space, the actions take values in the Hilbert space ℓ_+^2. Motivated by the perturbation theory of the NLS equation Korotyaev [47] conjectured – see also Korotyaev & Kuksin [49] – that the Hamiltonian $H^\star$ admits a real analytic extension to ℓ_+^2 and is strictly concave there. We prove this conjecture in a neighborhood of the origin in ℓ_+^2.

Theorem 18.3 *(i) The Hamiltonian $\mathcal{H}^\star$ admits a real analytic extension to FL_r^4, which is nonpositive and vanishes only at $\varphi = 0$.*

(ii) The Hamiltonian $H^\star$ admits a real analytic extension to the open neighborhood $\mathcal{V}^2$ contained in $\ell^2_{\mathbb{C}}$, which is nonpositive on $\ell^2_+ \cap \mathcal{V}^2$ and vanishes on $\ell^2_+ \cap \mathcal{V}^2$ only at $I = 0$. Moreover, it is strictly concave near $I = 0$ in the sense that for all I in a (sufficiently small) neighborhood of 0 in ℓ^2_+

$$\mathrm{d}_I^2 H^\star(J,J) \leqslant -\sum_{m\in\mathbb{Z}} J_m^2, \qquad \forall\ J = (J_m)_{m\in\mathbb{Z}} \in \ell^2. \quad \rtimes$$

Remark 18.4. In contrast to the frequency map $\omega^\star$, the Hamiltonian $H^\star$ does not admit a C^1-extension to any neighborhood U^q of the origin in ℓ^q_+ with $q > 2$. Indeed, this would imply that $\omega^\star(I) = \partial H^\star(I) \in \ell^{q'}$ for all I in U^q with $1/q + 1/q' = 1$. This, however, is impossible due to the diffeomorphism property of the frequencies established in Corollary 18.2. $\multimap$

Theorem 18.1 applies to study the solution map of the dNLS equation. To state this application, we first we need to introduce some more notation. According to [7] for any initial datum $\varphi \in H_r^s$, $s \geqslant 0$, there exists a unique, global in time solution $\psi(t,x) = \psi(t,x,\varphi)$ of (4.2), $\psi(\cdot,\cdot,\varphi) \in C(\mathbb{R}, H_r^s)$. In particular, for any time $t \in \mathbb{R}$, we have a nonlinear evolution operator $S_t = S(t,\cdot)\colon H_r^s \to H_r^s$ and, for any $T > 0$, a uniquely defined solution map

$$S\colon H_r^s \to C([-T,T], H_r^s), \qquad \varphi \mapsto \psi(\cdot,\cdot,\varphi), \tag{4.4}$$

which is analytic.

The appearance of the Hamiltonian $H_1 = \sum_{m\in\mathbb{Z}} I_m$ as a first order term in the expansion of the NLS frequencies (4.3) indicates that the dNLS solution map cannot be continuously extended to FL_r^p for $p > 2$. Instead, one consider the *renormalized NLS frequencies*

$$\omega_n^r = \omega_n - 4S_1 = (2n\pi)^2 + \omega_n^\star$$

corresponding to the Hamiltonian system

$$\mathrm{i}\partial_t \varphi = J\partial\mathcal{H}^r, \qquad \mathcal{H}^r = \mathcal{H} - 2\mathcal{H}_1^2.$$

When restricted to real type states $\varphi = (v,\bar{v})$, this Hamiltonian system admits the form of the *renormalized NLS equation* (dNLS)$_r$ also called *Wick-ordered NLS*

$$\mathrm{i}\partial_t v = \partial_{\bar{v}}\mathcal{H}^r = -\partial_x^2 v + 2|v|^2 v - 4\left(\int_{\mathbb{T}} |v|^2 \,\mathrm{d}x\right) v. \tag{4.5}$$

Since the dNLS flow preserves the L^2-norm of any initial datum on H_r^s, $s \geqslant 0$, we have for any dNLS solution $u(t)$ with initial datum u_0 that $v(t) = \mathrm{e}^{\mathrm{i}\|u_0\|_2 t} u(t)$ is a solution of (dNLS)$_r$ with the same initial datum. On H_r^s, $s \geqslant 0$, we thus can freely convert solutions of (4.2) into ones of (4.5) and vice versa, and for any $s \geqslant 0$ and $T > 0$ the (dNLS)$_r$ solution operator

$$S^r\colon H_r^s \to C([-T,T], H_r^s),$$

is analytic and equivalent to the solution operator of the dNLS equation S. In contrast to S, however, the solution operator S^r can be extended continuously, indeed analytically, to FL_r^p for any $2 < p < \infty$. To give a precise statement of our results we introduce the following.

Definition *A continuous curve $\gamma\colon (a,b) \to X$, $\gamma(0) = \varphi$, is called a* solution *of the (dNLS)$_r$ equation* (4.5) *in FL_r^p with initial datum φ, if and only if for any sequence of C^∞-potentials $(\varphi_k)_{k\geqslant 1}$, converging to φ in FL_r^p, the corresponding sequence $(S^r(t,\varphi_k))_{k\geqslant 1}$ of solutions of* (4.5) *with initial datum φ_k converges to $\gamma(t)$ in FL_r^p for any $t \in (a,b)$.* ⋊

Note that any solution according to this definition is necessarily unique. If it exists it will be denoted by $S^r(\cdot,\varphi)$.

Definition *The (dNLS)$_r$ equation* (4.5) *is said to be* locally in time C^ω-wellposed in FL_r^p *if and only if*

(a) for any initial datum $\varphi \in FL_r^p$ there exists a neighborhood U and a time $T > 0$ so that the initial value problem (4.5) *for any initial value $\psi \in U$ admits a solution $S(\cdot,\psi)$ in the aforementioned sense which is defined on the time interval $[-T,T]$, and*

(b) the solution map $S\colon U \to C([-T,T], FL_r^p)$ is real analytic.

For any open subset U of FL_r^p (dNLS)$_r$ is said to be globally in time C^ω-wellposed in U *if and only if*

(a') for any initial datum $\varphi \in U$ the initial value problem (4.5) *admits a solution $S^r(\cdot,\varphi)$ in the aforementioned sense which is defined globally in time, and*

(b') the solution map $S\colon U \to C([-T,T], FL_r^p)$ is real analytic for any $T > 0$. ⋊

We are now in a position to state our applications of Theorem 0.3 to the wellposedness of the dNLS and (dNLS)$_r$ equations in the Fourier Lebesgue spaces.

Theorem 18.5 *(i) (dNLS)$_r$ is globally in time C^ω-wellposed in FL_r^p, $1 < p \leqslant 2$.*

(ii) (dNLS)$_r$ is locally in time C^ω-wellposed in FL_r^p, $2 < p < \infty$.

(iii) (dNLS)$_r$ is globally in time C^ω-wellposed in a sufficiently small neighborhood of the origin in FL_r^p, $2 < p < \infty$.

(iv) dNLS is illposed in FL_r^p, $2 < p < \infty$, in the sense that for any $T > 0$ and any $q \geqslant p$ the solution map cannot be extended to a map $S\colon FL_r^p \to C([-T,T], FL_r^q)$ which is continuous at any point of $FL_r^p \setminus L_r^2$. ⋊

Related wellposedness results. Questions of wellposedness of nonlinear dispersive PDEs, such as dNLS, in spaces of low regularity have drawn a lot of interest recently – see for example the Dispersive Wiki [1] created by Tao. We just give a brief account on the results for dNLS which are close to Theorem 18.5. Molinet [57] showed that dNLS is ill-posed in H^s, $s < 0$, in the sense that the solution map $S_t\colon L_r^2 \to L_r^2$ is weakly discontinuous for any $t \neq 0$ at a carefully chosen point $u_0 \in L_r^2$. On the other hand, it was shown by Oh & Sulem [62] that the solution map of (dNLS)$_r$, $S^r\colon L_r^2 \to C([-T,T], L_r^2)$, is weakly continuous for any $T > 0$. Still, according to Burq et al. [8] the solution map S^r of (dNLS)$_r$ is not uniformly continuous on bounded subsets below L_r^2 – see also Christ et al. [12] and Kenig et al. [42]. When considered on the real line, the map $S_t\colon H^s(\mathbb{R}) \to H^s(\mathbb{R})$ is unbounded for any $t \neq 0$ and any $s < -1/2$ – see [12]. Christ, Holmer &

Tataru announced similar results on the circle. On the other hand, according to Koch & Tataru [43] the map $S_t : H^s(\mathbb{R}) \to H^s(\mathbb{R})$ is bounded for $-1/6 \leqslant s < 0$ and there exist weak solutions for any initial value $u_0 \in H^s(\mathbb{R})$ in this range of s which are not known to be unique. It is an open question whether $(\text{dNLS})_r$ is wellposed in H_r^s, $-1/2 \leqslant s < 0$, on the torus $\mathbb{T}$. A first positive result into this direction is due to Colliander & Oh [13], who established almost sure local wellposedness in H_r^s for $s > -1/3$ and almost sure global wellposedness in H_r^s for $s > -1/12$.

The Fourier Lebesgue spaces FL_r^p, $p > 2$, are another class of spaces in which the wellposedness theory of $(\text{dNLS})_r$ is considered. Christ [11] constructed local in time solutions of $(\text{dNLS})_r$ in FL_r^p for any $2 < p < \infty$ by the power series method using a »new method of solution«. More to the point, he shows that for any $2 < p < \infty$ and $R > 0$ there exists $\tau > 0$ so that the solution map admits a uniformly continuous extension $S^r : B_R(0) \subset FL_r^p \to C([0,\tau], FL_r^p)$. However, it is noted in [10] that this method of solution fails to provide uniqueness. Grünrock & Herr [25] obtained the same result by a fixed-point argument in Bourgain spaces, which provides uniqueness in the corresponding restriction norm spaces. It is also stated in [11], without proof, that the solutions for small initial data exist globally in time.

We contribute to the results of [11] and [25] in three ways. First, the solutions constructed in Theorem 18.5 depend analytically on the initial datum. We also prove that they exist globally in time for small initial values. Second, integrating the NLS flow in Birkhoff coordinates by quadrature is straightforward and shows that there is one, and only one, solution in the class $C([-T,T], FL_r^p)$ – see also Remark 22.3 at the end of Section 22. Finally, we show that dNLS is illposed in FL_r^p by failure of continuity of the solution map at *any* point of $FL_r^p \setminus L_r^2$.

19. The abelian integral F

In this section, we reconsider the quotient

$$\frac{\Delta^{\bullet}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} = -\mathrm{i} \prod_{m\in\mathbb{Z}} \frac{\lambda_m - \lambda}{w_m(\lambda)}, \tag{4.6}$$

introduced at the end of Section 10, and prove that it admits a globally defined analytic primitive F. For finite gap potentials this primitive turns out to be meromorphic and to admit a Laurent expansion whose coefficients are precisely the Hamiltonians of the NLS hierarchy. In following sections we derive from this expansion the novel formulas for the NLS frequencies as well as the NLS Hamiltonian.

By Lemma 10.11, for any $\varphi \in \mathcal{W}^p$, $1 < p < \infty$, the quotient (4.6) is analytic in both variables (λ, ψ) on $(\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} \overline{U_n}) \times V_\varphi$ and analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$. We proceed by defining on the same domain for any $n \in \mathbb{Z}$ the primitive

$$F_n(\lambda,\psi) \coloneqq \frac{1}{2}\left(\int_{\lambda_n^-(\psi)}^{\lambda} \frac{\Delta^{\bullet}(\mu,\psi)}{\sqrt[c]{\Delta^2(\mu,\psi) - 4}}\,\mathrm{d}\mu + \int_{\lambda_n^+(\psi)}^{\lambda} \frac{\Delta^{\bullet}(\mu,\psi)}{\sqrt[c]{\Delta^2(\mu,\psi) - 4}}\,\mathrm{d}\mu \right),$$

where the paths of integration are chosen to be admissible. These improper integrals exist, since for $\gamma_n = 0$ the integrand is analytic on U_n, while for $\gamma_n \neq 0$ it is of the form $1/\sqrt{\lambda_n^{\pm} - \lambda}$ locally around $\lambda_n^{\pm}$. Moreover, $\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^{\bullet}(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = 0$ by Lemma 10.11, hence the definition of F_n is independent of the chosen admissible path and one has

$$F_n(\lambda) = \int_{\lambda_n^-(\psi)}^{\lambda} \frac{\Delta^{\bullet}(\mu)}{\sqrt[c]{\Delta^2(\mu) - 4}}\,\mathrm{d}\mu = \int_{\lambda_n^+(\psi)}^{\lambda} \frac{\Delta^{\bullet}(\mu)}{\sqrt[c]{\Delta^2(\mu) - 4}}\,\mathrm{d}\mu.$$

Even though the eigenvalues $\lambda_n^\pm$ are, due to their lexicographical ordering, not even continuous on $\mathcal{W}^p$, the mappings F_n turn out to be analytic.

Lemma 19.1 *For every $\varphi \in \mathcal{W}^p$, $1 < p < \infty$, we have that*

(i) F_n is analytic in both variables (λ, ψ) on $(\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} \overline{U_m}) \times V_\varphi$ with gradient

$$\partial F_n(\lambda) = \frac{\partial\Delta(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}},$$

and $F_n(\lambda) \equiv F_n(\lambda, \varphi)$ is analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$.

(ii) $F_0(\lambda) = F_n(\lambda) - \mathrm{i}n\pi$ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$. In particular, F_0 extends continuously to all points $\lambda_m^\pm$, $m \in \mathbb{Z}$, and one has

$$F_0(\lambda_m^+) = F_0(\lambda_m^-) = -\mathrm{i}m\pi, \qquad m \in \mathbb{Z}.$$

(iii) locally uniformly in φ and uniformly as $|n| \to \infty$,

$$\sup_{\lambda \in G_n^+ \cup G_n^-} |F_n(\lambda)| = O(\gamma_n).$$

(iv) $F_n^2(\lambda)$ is analytic on $\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} G_m$ for every $n \in \mathbb{Z}$.

(v) If φ is of real type, then for any $\lambda \in G_n$

$$F_n(\lambda \pm \mathrm{i}0) = \pm \cosh^{-1} \frac{(-1)^n \Delta(\lambda)}{2}.$$

(vi) At the zero potential one has $F_n(\lambda, 0) = -\mathrm{i}\lambda + \mathrm{i}n\pi$. ⋊

Proof. (i) The proof of the analyticity of F_n on $(\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} \overline{U_m}) \times V_\varphi$ is standard but a bit technical and can be found in appendix F. The analyticity of $F_n(\lambda) = F_n(\lambda, \varphi)$ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$ follows immediately from the properties of $\frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}$ obtained in Lemma 10.11.

To obtain the formula for the gradient, we first consider the case of φ being of real type, Then $(-1)^n \Delta(\lambda, \varphi) \geqslant 2$ on G_n and hence

$$\min_{\lambda_n^- \leqslant \lambda \leqslant \lambda_n^+} (-1)^n \Delta(\lambda, \varphi) - \sqrt[+]{\Delta^2(\lambda, \varphi) - 4} > 0.$$

Thus, after possibly shrinking V_φ, we can choose a circuit Γ_n, which is contained in U_n, and an open neighborhood U_n' of Γ_n so that Γ_n circles around G_n, $\overline{U_n'} \subset U_m \setminus G_m$ for any potential in V_φ, and the real part of $(-1)^n \left(\Delta(\lambda, \psi) + \sqrt[c]{\Delta^2(\lambda, \psi) - 4}\right)$ is strictly positive on U_n'. In consequence, the principal branch of the logarithm

$$l_n(\lambda, \psi) = \log \frac{(-1)^n}{2} \left(\Delta(\lambda, \psi) + \sqrt[c]{\Delta^2(\lambda, \psi) - 4}\right)$$

is analytic on $U_n' \times V_\varphi$. Clearly, $\partial_\lambda l_n = \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}$ and $l_n(\lambda_n^\pm) \equiv 0$, hence $F_n = l_n$. Taking the gradient of the above identitiy yields on $U_n' \times V_\varphi$

$$\partial F_n = \partial l_n(\lambda) = \frac{\partial\Delta(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}}.$$

Since both hand sides of this identity are analytic in both variables on $(\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} \overline{U_m}) \times V_\varphi$ and analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$, the formula for the gradient extends to these domains by the identity theorem.

(ii) Note that $F_0(\lambda) = F_n(\lambda) + \int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda$. Clearly, $\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = 0$ for any $n \in \mathbb{Z}$ by Lemma 10.11, hence $F_n(\lambda_n^+) = F_n(\lambda_n^-) = 0$. Moreover,

$$\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = \sum_{k=0}^{n-1} \int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda, \quad n \geqslant 1,$$

while for $n \leqslant -1$

$$\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = -\sum_{k=n}^{-1} \int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda, \quad n \leqslant -1.$$

Therefore, it is to compute $\int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda$ for $k \in \mathbb{Z}$. To do this, first consider the case where φ is of real type. In this case $\mathrm{i}(-1)^k \sqrt[c]{\Delta^2(\lambda)-4} > 0$ for $\lambda_k^+ < \lambda < \lambda_{k+1}^-$ – c.f. [23, Section 5] – so

$$\int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda = \mathrm{i}(-1)^k \int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[+]{4-\Delta^2(\lambda)}}\,\mathrm{d}\lambda = \mathrm{i}(-1)^k \sin^{-1}\frac{\Delta(\lambda)}{2}\bigg|_{\lambda_k^+}^{\lambda_{k+1}^-} = -\mathrm{i}\pi,$$

and hence $\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[+]{4-\Delta^2(\lambda)}}\,\mathrm{d}\lambda = -\mathrm{i}n\pi$ for any $n \in \mathbb{Z}$. The function $\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[+]{4-\Delta^2(\lambda)}}\,\mathrm{d}\lambda = F_0(\lambda) - F_n(\lambda)$ is analytic on V_φ by item (i), therefore, $\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[+]{4-\Delta^2(\lambda)}}\,\mathrm{d}\lambda = -\mathrm{i}n\pi$ holds true on all of V_φ in view of Lemma E.2.

(iv) In view of item (i) it remains to show that F_n^2 admits also for $\gamma_n \neq 0$ an analytic extension from $U_n \setminus G_n$ to all of U_n. Write (2.15) in the form

$$\frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} = -\mathrm{i}\frac{\lambda_n^\bullet - \lambda}{w_n(\lambda)}\chi_n(\lambda), \qquad \chi_n(\lambda) = \prod_{m\neq n} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}. \tag{4.7}$$

The functionals χ_n, $n \in \mathbb{Z}$, are analytic on U_n by Corollary 10.6. Moreover, the roots $w_n(\lambda)$, $n \in \mathbb{Z}$, admit opposite signs on opposite sides of G_n – see (2.12). Therefore, in view of $F_n(\lambda) = \int_{\lambda_n^+}^{\lambda} \frac{\Delta^\bullet(\mu)}{\sqrt[c]{\Delta^2(\mu)-4}}\,\mathrm{d}\mu$, for any $\lambda \in G_n$,

$$F_n\big|_{G_n^+}(\lambda) = -F_n\big|_{G_n^-}(\lambda).$$

Consequently, F_n^2 is continuous and hence analytic on all of U_n.

(iii) If $\gamma_n = 0$, then $G_n = \{\lambda_n^\pm\}$ and $F(\lambda_n^\pm) = 0$ so there is nothing to show. Thus suppose $\gamma_n \neq 0$. We have $\lambda_n^\bullet = \tau_n + \gamma_n^2\ell_n^p$ and $\sup_{\lambda\in U_n}|\chi_n(\lambda) - 1| = \ell_n^p$ locally uniformly on V_φ by Lemma 10.10 and Lemma 10.6. In view of (4.7) it follows with Lemma 10.4 that

$$\sup_{\lambda\in G_n^-\cup G_n^+} |F_n(\lambda)| \leqslant O\left(\sup_{\lambda\in G_n} |\lambda_n^\bullet - \lambda|\right) = O(\gamma_n),$$

uniformly on V and for all $n \in \mathbb{Z}$.

(v) If φ is of real type, then for any $\lambda \in G_n$

$$F_n(\lambda \pm \mathrm{i}0) = \pm\int_{\lambda_n^-}^{\lambda} \frac{(-1)^n\Delta^\bullet(\mu)}{\sqrt[+]{\Delta^2(\mu)-4}}\,\mathrm{d}\mu = \pm\cosh^{-1}\frac{(-1)^n\Delta(\lambda)}{2}.$$

(vi) At the zero potential, $\frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\Big|_{\varphi=0} = -\mathrm{i}$, which follows directly from the product representation (2.15). Integration thus yields $F_n(\lambda, 0) = -\mathrm{i}(\lambda - n\pi)$. ∎

To simplify notation we write $F(\lambda) \equiv F_0(\lambda)$ and note that $F(\lambda) = F_n(\lambda) - \mathrm{i}n\pi$ for any $n \in \mathbb{Z}$.

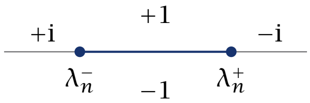

Figure 4.1.: Signs of $F_n(\lambda)$ around G_n.

Lemma 19.2 *Suppose φ is a finite gap potential of real type, then F is analytic outside a disc of finite radius centered at the origin and admits the Laurent expansion*

$$F(\lambda) = -\mathrm{i}\lambda + \mathrm{i}\sum_{n\geqslant 1}\frac{\mathcal{H}_n}{(2\lambda)^n}, \tag{4.8}$$

where $\mathcal{H}_n$ denotes the nth Hamiltonian of the NLS hierarchy. $\rtimes$

Proof. By the preceding lemma $F(\lambda)$ is analytic on $\mathbb{C}\setminus\bigcup_{\gamma_m\neq 0} G_m$. Suppose φ is a finite gap potential, then $F(\lambda)$ is analytic outside a disc of finite radius. Moreover, the product expansion (2.15) of $\frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}$ is finite, whence one verifies directly that $F(\lambda) = O(\lambda)$ uniformly as $|\lambda| \to \infty$. Therefore, F is meromorphic with a pole at infinity of order at most one, and it suffices to determine the Laurent expansion of F along an arbitrary sequence ν_n with $|\nu_n| \to \infty$.

Since φ is assumed to be of real type, the function $(-1)^{n+1}\Delta(\lambda)$, for any $n \in \mathbb{Z}$, is strictly increasing from -2 to 2 on $[\lambda_n^+, \lambda_{n+1}^-]$ and the canonical root for $\lambda_n^+ < \lambda < \lambda_{n+1}^-$ is given by

$$\sqrt[c]{\Delta^2(\lambda) - 4} = (-1)^{n+1}\mathrm{i}\sqrt[+]{4 - \Delta^2(\lambda)}.$$

Furthermore, one computes for $\lambda_n^+ < \lambda < \lambda_{n+1}^-$ that

$$\partial_\lambda\left(-\mathrm{i}\sin^{-1}\left((-1)^{n+1}\frac{\Delta(\lambda)}{2}\right)\right) = \mathrm{i}\frac{(-1)^n\Delta^\bullet(\lambda)/2}{\sqrt[+]{1-\Delta^2(\lambda)/4}} = \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}.$$

Hence for any $\lambda_n^+ \leqslant \lambda \leqslant \lambda_{n+1}^-$,

$$\begin{aligned}
F(\lambda) &= F(\lambda_n^+) + \int_{\lambda_n^+}^{\lambda}\frac{\Delta^\bullet(\mu)}{\sqrt[c]{\Delta^2(\mu)-4}}\,\mathrm{d}\mu \\
&= -\mathrm{i}n\pi + \left[-\mathrm{i}\sin^{-1}\left((-1)^{n+1}\frac{\Delta(\mu)}{2}\right)\right]_{\lambda_n^+}^{\lambda} \\
&= -\mathrm{i}(n+1/2)\pi - \mathrm{i}\sin^{-1}\left((-1)^{n+1}\frac{\Delta(\lambda)}{2}\right).
\end{aligned}$$

Let $\nu_n = (n+1/2)\pi$, then by [23, Theorem 4.8][1] for any $N \geqslant 1$

$$\Delta(\nu_n) = 2\cos\mathrm{i}\sigma_N(\nu_n) + O(\nu_n^{-N}), \qquad \sigma_N(\lambda) = -\mathrm{i}\lambda + \mathrm{i}\sum_{n=1}^{N}\frac{\mathcal{H}_n}{(2\lambda)^n},$$

as $|n| \to \infty$. Using that $\partial_z\sin^{-1}(z) = 1/\sqrt[+]{1-z^2}$ one gets by the mean value theorem

$$\left|\sin^{-1}\left((-1)^{n+1}\frac{\Delta(\nu_n)}{2}\right) - \sin^{-1}\left((-1)^{n+1}\cos\mathrm{i}\sigma_N(\nu_n)\right)\right| = O(\nu_n^{-N}),$$

and hence

$$F(\nu_n) = -\mathrm{i}\nu_n - \mathrm{i}\sin^{-1}\left((-1)^{n+1}\cos\mathrm{i}\sigma_N(\nu_n)\right) + O(\nu_n^{-N}), \qquad n \to \infty.$$

[1]Note that in in comparison to [23] we multiplied for $n \geqslant 2$ the nth Hamiltonian with the factor $(-\mathrm{i})^{n+1}$.

Finally, writing $(-1)^{n+1} = -\sin\nu_n$ one gets by the addition theorem for the sine

$$(-1)^{n+1}\cos \mathrm{i}\sigma_N(\nu_n) = \sin(\mathrm{i}\sigma_N(\nu_n) - \nu_n),$$

and hence

$$-\mathrm{i}\sin^{-1}\Big((-1)^{n+1}\cos \mathrm{i}\sigma_N(\nu_n)\Big) = \sigma_N(\nu_n) + \mathrm{i}\nu_n.$$

This gives $F(\nu_n) = \sigma_N(\nu_n) + O(\nu_n^{-N})$, hence the Laurent coefficients of F can be determined from σ_N. ∎

The following expansion will be of use later.

Corollary 19.3 *Suppose φ is a finite gap potential of real type, then F^3 is analytic outside a disc of finite radius and admits the Laurent expansion*

$$F^3(\lambda) = \mathrm{i}\lambda^3 - \mathrm{i}\frac{3}{2}\mathcal{H}_1\lambda - \mathrm{i}\frac{3}{4}\mathcal{H}_2 - \mathrm{i}\frac{3}{8}\frac{\mathcal{H} - 2\mathcal{H}_1^2}{\lambda} + O(\lambda^{-2}). \qquad \rtimes$$

We conclude this section by refining the asymptotics $\sup_{\lambda\in G_n^+\cup G_n^-}|F_n(\lambda)| = O(\gamma_n)$ of Lemma 19.1.

Lemma 19.4 *For any $1 < p < \infty$, locally uniformly $\mathcal{W}^p$*

$$\sup_{\lambda\in G_n^+\cup G_n^-}|F_n(\lambda) - \mathrm{i}w_n(\lambda)| = \gamma_n(\ell_n^{p/2} + \ell_n^{1+}).$$

At the zero potential $F_n(\lambda) = \mathrm{i}w_n(\lambda) = -\mathrm{i}\lambda + \mathrm{i}n\pi$ holds without the error term. $\rtimes$

Proof. With (4.7) write F_n in the form

$$F_n(\lambda) = \int_{\lambda_n^-}^{\lambda}\frac{\Delta^\bullet(\mu)}{\sqrt[c]{\Delta^2(\mu)-4}}\,\mathrm{d}\mu = -\mathrm{i}\int_{\lambda_n^-}^{\lambda}\frac{\lambda_n^\bullet - \mu}{w_n(\mu)}\chi_n(\mu)\,\mathrm{d}\mu, \qquad \chi_n(\lambda) = \prod_{m\neq n}\frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}.$$

By Lemma 10.10, $\lambda_n^\bullet - \tau_n = \gamma_n^2\ell_n^p$, hence Lemma 10.8 gives $\sup_{\lambda\in U_n}|\chi_n(\lambda) - 1| = \ell_n^{p/2} + \ell_n^{1+}$. As an immediate consequence we obtain from Lemma 10.4 that

$$\sup_{\lambda\in G_n^+\cup G_n^-}\left|F_n(\lambda) - (-\mathrm{i})\int_{\lambda_n^-}^{\lambda}\frac{\lambda_n^\bullet - \mu}{w_n(\mu)}\,\mathrm{d}\mu\right| \leqslant \max_{\lambda\in G_n^+\cup G_n^-}|(\lambda_n^\bullet - \lambda)(\chi_n(\lambda) - 1)| = \gamma_n(\ell_n^{p/2} + \ell_n^{1+}).$$

One further checks that $\partial_\lambda w_k(\lambda) = -\frac{\tau_k - \lambda}{w_k(\lambda)}$ for $\lambda \notin G_n$, hence

$$-\mathrm{i}\int_{\lambda_k^-}^{\lambda}\frac{\lambda_k^\bullet - \xi}{w_k(\xi)}\,\mathrm{d}\xi = \mathrm{i}w_k(\lambda) + \mathrm{i}(\tau_k - \lambda_k^\bullet)\int_{\lambda_k^-}^{\lambda}\frac{1}{w_k(\xi)}\,\mathrm{d}\xi.$$

If $\gamma_k = 0$, then $\tau_k = \lambda_k^\bullet$ and the claim is evident. On the other hand, if $\gamma_k \neq 0$, then Lemma 10.4 gives $\sup_{\lambda\in G_k^-\cup G_k^+}\left|\int_{\lambda_k^-}^{\lambda}\frac{1}{w_k(\xi)}\,\mathrm{d}\xi\right| \leqslant \pi$ and the claim follows with the estimate $\tau_k - \lambda_k^\bullet = \gamma_k^2\ell_k^p$. ∎

20. NLS frequencies

In this section we derive a novel formula for the NLS frequencies which we then use to study their decay properties. The frequencies can be viewed either as analytic functionals of the potential φ on $\mathcal{W}^p$ or as analytic functionals of the actions $I = (I_m)_{m\in\mathbb{Z}}$ on the complex neighborhood $\mathcal{V}^{p/2}$ of $\ell_+^{p/2}$ introduced in Section 18. Which case is at hand should be always clear from the context, hence we do not introduce different notations for them. Our starting point is the following identity for the nth NLS frequency

$$\omega_n = \{\theta_n, \mathcal{H}\},$$

which a priori holds on $H^1_c \cap (\mathcal{W}^2 \setminus Z_n)$, where $Z_n = \{\varphi \in \mathcal{W}^2 : \gamma_n^2(\varphi) = 0\}$ is the analytic subvariety of $\mathcal{W}^2$ introduced in Chapter 2.

It turns out to be convenient to introduce for any integers $n, k \in \mathbb{Z}$ and $m \geqslant 0$ the moments

$$\Omega_{nk}^{(m)} \coloneqq \int_{\Gamma_k} \frac{F_k^m(\lambda)\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}\,\mathrm{d}\lambda,$$

with the canonical root introduced in Section 10, the functions ψ_n constructed in Section 12, and the Abelian integral F_k defined in Section 19. We recall from Section 12 the product representation of the quotient $\psi_n(\lambda)/\sqrt[c]{\Delta^2(\lambda)-4}$, obtained from (2.20), (2.13), and (2.25),

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} = \frac{-\mathrm{i}}{w_n(\lambda)}\zeta_n(\lambda) = \frac{\sigma_k^n-\lambda}{w_k(\lambda)}\zeta_k^n(\lambda). \tag{4.9}$$

Here we have set $\sigma_n^n = \tau_n$ and for any $k \in \mathbb{Z}$ the functions ζ_k and ζ_k^n are analytic on U_k,

$$\zeta_k^n(\lambda) = \frac{-\mathrm{i}}{\tau_n-\lambda}\zeta_k(\lambda), \qquad \zeta_k(\lambda) = -\prod_{m\neq k}\frac{\sigma_m^n-\lambda}{w_m(\lambda)}. \tag{4.10}$$

Lemma 20.1 *(i)* $\Omega_{nk}^{(0)} = 2\pi\delta_{nk}$, *for all* $n, k \in \mathbb{Z}$. ⋊

(ii) Each moment $\Omega_{nk}^{(m)}$, $n, k \in \mathbb{Z}$, $m \geqslant 1$, *is analytic on* $\mathcal{W}^p$, $1 < p < \infty$.

(iii) $\Omega_{nk}^{(2l+1)} = 0$, *for all* $n, k \in \mathbb{Z}$ *and* $l \geqslant 0$.

(iv) If $\gamma_k = 0$, *then* $\Omega_{nk}^{(m)} = 0$, *for all* $n \in \mathbb{Z}$ *and* $m \geqslant 1$.

Proof. (i): The identity follows from the characterization (2.21) of the functions ψ_n.

(ii): Let $\varphi \in \mathcal{W}^p$. As in Section 10, we choose circuits Γ_k, $k \in \mathbb{Z}$, and open neighborhoods U_k' of Γ_k such that Γ_k circles around G_k and $\overline{U_k'} \subset U_k \setminus G_k$ for any potential in V_φ. In view of Lemma 10.7, Theorem 12.1, and Lemma 19.1, the integrand $\frac{F_k^m(\lambda)\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}$ is analytic on $U_k' \times V_\varphi$ for any $k \in \mathbb{Z}$. Consequently, $\Omega_{nk}^{(m)}$ is analytic on V_φ.

(iii): By Lemma 19.1 and (2.14), the function F_k and the canonical root both extend continuously to the opposite sides $G_k^\pm$ of the gap G_k and take opposite signs there. Consequently, for any $l \geqslant 0$ the quotient $F_k^{2l+1}(\lambda)/\sqrt[c]{\Delta^2(\lambda)-4}$ extends continuously from $U_k \setminus G_k$ to U_k and hence is analytic on all of U_k. Together with the fact that ψ_n is an entire function, we conclude $\Omega_{nk}^{(2l+1)} = 0$ for all $n, k \in \mathbb{Z}$.

(iv): In view of item (iii) it remains to consider the case $m = 2l$, $l \geqslant 1$, and $n, k \in \mathbb{Z}$ with $\gamma_k = 0$. Suppose $k \neq n$. Since $\sigma_k^n = \tau_k$ by Theorem 12.1, and $w_k(\lambda) = \tau_k - \lambda$ in view of (2.9), by (4.9) the quotient $\psi_n(\lambda)/\sqrt[c]{\Delta^2(\lambda)-4}$ equals $\zeta_k^n(\lambda)$ and hence is analytic on U_k. Since by Lemma 19.1 also F_k^{2l} is analytic on U_k, we conclude $\Omega_{nk}^{(2l)} = 0$. Now suppose $k = n$. Since ζ_n and F^{2l} are analytic on U_n and $w_n(\lambda) = \tau_n - \lambda$, we have in view of (4.9) and Cauchy's Theorem that $\Omega_{nn}^{(2)} = 2\pi F_n^{2l}(\tau_n)\zeta_n(\tau_n)$. Since $\gamma_n = 0$ we find by Lemma 19.1 that $F_n(\tau_n) = F_n(\lambda_n^\pm) = 0$ proving the claim. ∎

Lemma 20.2 *For any finite gap potential of real type and any $n \in \mathbb{Z}$*

$$\omega_n^\star = \omega_n - 4\mathcal{H}_1 - (2n\pi)^2 = -\frac{4}{2\pi}\sum_{k\in\mathbb{Z}} \Omega_{nk}^{(2)}. \quad \rtimes \tag{4.11}$$

Proof. Suppose φ is a finite gap potential, meaning the set $A = \{k \in \mathbb{Z} : \gamma_k(\varphi) \neq 0\}$ is finite. By Corollary 19.3, the function $F^3(\lambda)$ is analytic outside a sufficiently large circle C_r enclosing all open gaps G_k, $k \in A$, and admits the Laurent expansion

$$F^3(\lambda) = \mathrm{i}\lambda^3 - \mathrm{i}\frac{3}{2}\mathcal{H}_1\lambda + \frac{3}{4}\mathcal{H}_2 - \mathrm{i}\frac{3}{8}\frac{\mathcal{H} - 2\mathcal{H}_1^2}{\lambda} + O(\lambda^{-2}).$$

Therefore, by Cauchy's Theorem

$$\mathcal{H} - 2\mathcal{H}_1^2 = \frac{8}{6\pi}\int_{C_r} F^3(\lambda)\,\mathrm{d}\lambda.$$

Consider any $n \in \mathbb{Z}$ with $\gamma_n(\varphi) \neq 0$. Then $\theta_n \mod \pi$ is analytic near φ by Theorem 13.1 and one has

$$\omega_n - 4\mathcal{H}_1 = \{\theta_n, \mathcal{H} - 2\mathcal{H}_1^2\} = -\frac{8}{6\pi}\int_{C_r}\{F^3(\lambda), \theta_n\}\,\mathrm{d}\lambda = -\frac{8}{2\pi}\int_{C_r}\frac{F^2(\lambda)\{\Delta(\lambda), \theta_n\}}{\sqrt[c]{\Delta^2(\lambda) - 4}}\,\mathrm{d}\lambda.$$

Using $2\{\Delta(\lambda), \theta_n\} = \psi_n(\lambda)$ – c.f. [33, Lemma 18.2] – we thus obtain

$$\omega_n - 4\mathcal{H}_1 = -\frac{4}{2\pi}\int_{C_r}\frac{F^2(\lambda)\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}}\,\mathrm{d}\lambda.$$

First, $F^2(\lambda)$ is analytic on $\mathbb{C}\setminus\bigcup_{k\in A} G_k$. Second, for any $k \in \mathbb{Z}\setminus S$ one has by the same arguments as in the proof of Lemma 20.1 (iii) that the quotient $\psi_n(\lambda)/\sqrt[c]{\Delta^2(\lambda) - 4}$ is analytic on U_k. Consequently, the integrand is analytic on $\mathbb{C} \setminus \bigcup_{k\in A} G_k$ and one obtains by contour deformation

$$\omega_n - 4\mathcal{H}_1 = -\frac{4}{2\pi}\sum_{k\in A}\int_{\Gamma_k}\frac{F^2(\lambda)\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}}\,\mathrm{d}\lambda.$$

Proceeding by expanding $F(\lambda)^2 = (F_k(\lambda) - \mathrm{i}k\pi)^2 = F_k^2(\lambda) - 2\mathrm{i}(k\pi)F_k(\lambda) - (k\pi)^2$ and using that $\Omega_{nk}^{(1)} \equiv 0$ by Lemma 20.1, we thus get

$$\omega_n - 4\mathcal{H}_1 = -\frac{4}{2\pi}\sum_{k\in A}\left(\Omega_{nk}^{(2)} - (k\pi)^2\Omega_{nk}^{(0)}\right) = \sum_{k\in\mathbb{Z}}\left(-\frac{4}{2\pi}\Omega_{nk}^{(2)} + (2k\pi)^2\delta_{kn}\right).$$

Here, we used in the second statement that $\Omega_{nk}^{(2)} = 0$ for all $k \in \mathbb{Z} \setminus S$. This shows that (4.11) holds for all n with $\gamma_n(\varphi) \neq 0$.

Now consider any $n \in \mathbb{Z}$ with $\gamma_n(\varphi) = 0$. We can choose a sequence of finite gap potentials φ_l in H_r^1 with $\gamma_k(\varphi_l) = \gamma_k(\varphi)$ for $k \neq n$, $\gamma_n(\varphi_l) \neq 0$, and $\varphi_l \to \varphi$ in H_r^1. In particular, $A^{(l)} \equiv A(\varphi^{(l)}) \coloneqq \{j \in \mathbb{Z} : \gamma_j(\varphi_l) \neq 0\}$ is given by $A \cup \{n\}$ for any $l \geqslant 1$. Since by Lemma 20.1 each $\Omega_{nk}^{(2)}$, $k \in \mathbb{Z}$, is continuous, indeed analytic, on $\mathcal{W}^p$, and A is finite and independent of l, it follows that

$$\sum_{k\in\mathbb{Z}}\Omega_{nk}^{(2)}(\varphi_l) = \sum_{k\in A\cup\{n\}}\Omega_{nk}^{(2)}(\varphi_l) \to \sum_{k\in A\cup\{n\}}\Omega_{nk}^{(2)}(\varphi) = \sum_{k\in\mathbb{Z}}\Omega_{nk}^{(2)}(\varphi).$$

On the other hand, ω_n is continuous at $\varphi \in H_r^1$ which shows that (4.11) holds for all $n \in \mathbb{Z}$. ∎

We proceed by deriving decay estimates for $\Omega_{nk}^{(2)}$.

Lemma 20.3 *Locally uniformly on $\mathcal{W}^p$, $1 < p < \infty$, and uniformly in n*

$$\Omega^{(2)}_{nk} = \frac{\gamma_k^3(\ell_k^{p/2}[n] + \ell_k^{1+}[n])}{n-k}, \quad k \neq n, \qquad \Omega^{(2)}_{nn} = \frac{\gamma_n^2}{4}\left(\pi + \ell_n^{p/2} + \ell_n^{1+}\right).$$

In more detail, there exists a sequence α_k^n so that $\Omega^{(2)}_{nk} = \frac{\gamma_k^3}{n-k}\alpha_k^n$ so that for any $r > 1$ with $r \geqslant p/2$

$$\sum_{k \neq n} |\alpha_k^n|^r \leqslant C_r,$$

and $C_r > 0$ can be chosen uniformly in n and locally uniformly on $\mathcal{W}^p$. ⋊

Proof. If $\gamma_k = 0$, then $\Omega^{(2)}_{nk} = 0$ by Lemma 20.1, hence it remains to consider the case where $\gamma_k \neq 0$. We begin with the the case $k \neq n$. Using the representation (4.9), shrinking the contour of integration Γ_k to $G_k^- \cup G_k^+$, and using (2.12) gives

$$\Omega^{(2)}_{nk} = 2\int_{G_k^-} \frac{F_k^2(\lambda)(\sigma_k^n - \lambda)\zeta_k^n(\lambda)}{w_k(\lambda)}\,\mathrm{d}\lambda.$$

Since $\sigma_k^n = \tau_k + \gamma_k^2\ell_k^p[n]$ by Theorem 12.1, it follows from Lemma 10.8 and (2.26) that

$$(n-k)\zeta_k^n(\lambda)\Big|_{U_k} = \frac{\mathrm{i}}{\pi} + \ell_k^{p/2}[n] + \ell_k^{1+}[n].$$

Moreover, by Lemma 19.4 uniformly on $G_k^\pm$,

$$F_k(\lambda)^2 = -w_k^2(\lambda) + \gamma_k^2(\ell_k^{p/2} + \ell_k^{1+}).$$

Combining both expansions, then yields

$$(n-k)\Omega^{(2)}_{nk} = 2\int_{G_k^-} \frac{\left(-w_k^2(\lambda) + \gamma_k^2(\ell_k^{p/2} + \ell_k^{1+})\right)\left(\sigma_k^n - \lambda\right)\left(\mathrm{i}/\pi^2 + \ell_k^{p/2}[n] + \ell_k^{1+}[n]\right)}{w_k(\lambda)}\,\mathrm{d}\lambda.$$

Since $\max_{\lambda\in G_n}|\sigma_k^n - \lambda| = O(\gamma_k)$ and $\max_{\lambda\in G_n^+\cup G_n^-}|w_k(\lambda)| = |\gamma_k|/2$, Lemma 10.4 further shows

$$(n-k)\Omega^{(2)}_{nk} = -\mathrm{i}\frac{2}{\pi^2}\int_{G_k^-} w_k(\lambda)(\sigma_k^n - \lambda)\,\mathrm{d}\lambda + \gamma_k^3(\ell_k^{p/2}[n] + \ell_k^{1+}[n]).$$

Using (2.12) one also computes that

$$\begin{aligned}
\int_{G_k^-} w_k(\lambda)\,\mathrm{d}\lambda &= \mathrm{i}\frac{\gamma_k^2}{4}\int_{-1}^{1} \sqrt[+]{1-t^2}\,\mathrm{d}t = \mathrm{i}\pi\frac{\gamma_k^2}{8},\\
\int_{G_k^-} (\tau_k - \lambda)w_k(\lambda)\,\mathrm{d}\lambda &= -\mathrm{i}\frac{\gamma_k^3}{8}\int_{-1}^{1} t\sqrt[+]{1-t^2}\,\mathrm{d}t = 0,
\end{aligned}$$

hence, together with $\sigma_k^n - \tau_k = \gamma_k^2\ell_k^p[n]$, we conclude

$$\int_{G_k^-} w_k(\lambda)(\sigma_k^n - \lambda)\,\mathrm{d}\lambda = \mathrm{i}\pi\frac{\gamma_k^2}{8}(\sigma_k^n - \tau_k) = \gamma_k^4\ell_k^p[n].$$

Altogether we thus find for $k \neq n$

$$\Omega^{(2)}_{nk} = \frac{\gamma_k^3(\ell_k^{p/2}[n] + \ell_k^{1+}[n])}{n-k}.$$

If $k = n$, then (4.9) has the form

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} = -\frac{\mathrm{i}}{w_n(\lambda)}\zeta_n(\lambda),$$

where $\zeta_n(\lambda) = -1 + \ell_n^{p/2} + \ell_n^{1+}$, $\lambda \in U_n$. Consequently, by the same arguments as above

$$\begin{aligned}
\Omega^{(2)}_{nn} &= -\mathrm{i}2\int_{G_n^-} \frac{(-w_n^2(\lambda) + \gamma_n^2(\ell_n^{p/2} + \ell_n^{1+}))(-1 + \ell_n^{p/2} + \ell_n^{1+})}{w_n(\lambda)}\,\mathrm{d}\lambda\\
&= -\mathrm{i}2\int_{G_n^-} w_n(\lambda)\,\mathrm{d}\lambda + \gamma_n^2(\ell_n^{p/2} + \ell_n^{1+})\\
&= \frac{\pi}{4}\gamma_n^2 + \gamma_n^2(\ell_n^{p/2} + \ell_n^{1+}) = \frac{\gamma_n^2}{4}\left(\pi + \ell_n^{p/2} + \ell_n^{1+}\right). \qquad \blacksquare
\end{aligned}$$

The decay estimates of the moments $\Omega_{nk}^{(2)}$ together with formula (4.11) for finite gap potentials of real type allow us to establish the analytic extension of the frequencies $\omega_n^\star$.

Theorem 20.4 *For any $n \in \mathbb{Z}$, the sum $-\frac{4}{2\pi}\sum_{k\in\mathbb{Z}}\Omega_{nk}^{(2)}$ converges absolutely and locally uniformly on $\mathcal{W}^p$, $1 < p < \infty$, to the real analytic function $\omega_n^\star$,*

$$\omega_n^\star = -\frac{4}{2\pi}\sum_{k\in\mathbb{Z}}\Omega_{nk}^{(2)}. \quad \rtimes$$

Proof. Each moment $\Omega_{nk}^{(2)}$ is real analytic on $\mathcal{W}^p$ by Lemma 20.1. Further, in view of Lemma 20.3 for any $k \neq n$

$$\Omega_{nk}^{(2)} = \frac{\gamma_k^3}{n-k}(\ell_k^{p/2}[n] + \ell_k^{1+}[n]),$$

locally uniformly on $\mathcal{W}^p$. Thus the sum $\Omega_n^{(2)} = -\frac{4}{2\pi}\sum_{k\in\mathbb{Z}}\Omega_{nk}^{(2)}$ is absolutely and locally uniformly convergent to an analytic function on $\mathcal{W}^p$. Moreover, the identity $\omega_n^\star = \Omega_n^2$, $n \in \mathbb{Z}$, holds for any real valued finite gap potential by Lemma 20.2. Consequently, $\Omega_n^{(2)}$ is the unique analytic extension of $\omega_n^\star$ from the set of finite gap potentials to $\mathcal{W}^p$. ∎

Our second main result for the frequencies $\omega_n^\star$ concerns their asymptotic behavior. To this end, we introduce frequency map $\omega^\star = (\omega_n^\star)_{n\in\mathbb{Z}}$.

Theorem 20.5 *(i) The map $\omega^\star : \mathcal{W}^2 \to \ell_\mathbb{C}^r$ is real-analytic for any $r > 1$.*

(ii) For any $p > 2$ the map $\omega^\star$ admits an analytic extension $\omega^\star : \mathcal{W}^p \to \ell_\mathbb{C}^{p/2}$ which satisfies

$$\omega_n^\star + 2\frac{\gamma_n^2}{4} = \ell_n^{p/3} + \ell_n^{1+}$$

locally uniformly on $\mathcal{W}^p$. $\rtimes$

Proof. By Lemma 20.3 we have $\Omega_{nk}^{(2)} = \frac{\gamma_k^3}{n-k}a_k^n$ for $k \neq n$, with

$$\left(\sum_{k\neq n}|a_k^n|^r\right)^{1/r} \leqslant C, \qquad \begin{cases} r = p/2, & \text{if } p > 2, \\ \forall\ r > 1, & \text{if } p = 2, \end{cases}$$

and the constant C can be chosen uniformly in n. Choose r' so that $1/r + 1/r' = 1$. Then applying Hölder's and subsequently Young's inequality gives for any $s \geqslant r'$

$$\sum_{n\in\mathbb{Z}}\left|\sum_{k\neq n}\Omega_{nk}^{(2)}\right|^s \leqslant C^s\sum_{n\in\mathbb{Z}}\left|\sum_{k\neq n}\frac{|\gamma_k|^{3r'}}{|n-k|^{r'}}\right|^{s/r'} \leqslant C^s\left(\sum_{m\neq 0}\frac{1}{|m|^{r'}}\right)^{s/r'}\left(\sum_{k\in\mathbb{Z}}|\gamma_k|^{3s}\right),$$

and we need $s \geqslant p/3$ for the right hand side to be finite. On the other hand, for $s < r'$ we apply the basic inequality $(|a| + |b|)^{s/r'} \leqslant |a|^{s/r'} + |b|^{s/r'}$ to get

$$\sum_{n\in\mathbb{Z}}\left|\sum_{k\neq n}\Omega_{nk}^{(2)}\right|^s \leqslant C^s\sum_{n\in\mathbb{Z}}\left|\sum_{k\neq n}\frac{|\gamma_k|^{3r'}}{|n-k|^{r'}}\right|^{s/r'} \leqslant C^s\left(\sum_{m\neq 0}\frac{1}{|m|^{s}}\right)\left(\sum_{k\in\mathbb{Z}}|\gamma_k|^{3s}\right),$$

and one needs $s > 1$ and $s \geqslant p/3$ for the right hand side to be finite. We thus conclude that

$$\sum_{k\neq n}\Omega_{nk}^{(2)} = \ell_n^{p/3} + \ell_n^{1+}.$$

On the other hand, $\Omega_{nn}^{(2)} = \frac{\gamma_n^2}{4}(\pi + \ell_n^{p/2}) = \ell_n^{p/2}$. Consequently, for any $p \geqslant 2$

$$\omega_n^\star = -2\frac{\gamma_n^2}{4} + \ell_n^{p/3} + \ell_n^{1+},$$

from which all the claims follow. ∎

Proof of Theorem 18.1. By the same arguments as in the proof of Theorem 20.3 in [23], one concludes from item (i) of Theorem 20.5 that $\omega^\star$, viewed as a function of the actions, is a real analytic function $\omega^\star\colon \ell^1_+ \to \ell^r$ for any $r > 1$, and from item (ii) that for any $p > 2$, $\omega^\star$ extends to a real analytic function $\omega^\star\colon \mathcal{V}^{p/2} \to \ell^{p/2}$. Here $\mathcal{V}^{p/2}$ denotes the image of the map $\mathcal{W}^p \to \ell^{p/2}_{\mathbb{C}}$, $\varphi \mapsto (I_m)_{m\in\mathbb{Z}}$, and is an open subset of $\ell^{p/2}_{\mathbb{C}}$ which contains ℓ^1_+. Since locally uniformly on $\mathcal{W}^p$ by Theorem 11.2

$$\frac{\gamma_n^2}{4} = I_n + \ell^{p/4}_n + \ell^{1+}_n,$$

item (ii) of Theorem 20.5 implies that

$$\omega^\star_n + 2I_n = \ell^{p/3}_n + \ell^{1+}_n. \quad \blacksquare \tag{4.12}$$

Proof of Corollary 18.2. (i): Near $I = 0$ the frequency $\omega^\star_n$ admits by (4.3) the Taylor expansion

$$\omega^\star_n = -2I_n + \cdots .$$

In particular, for any $p > 2$, $\omega^\star\colon \mathcal{V}^{p/2} \to \ell^{p/2}_{\mathbb{C}}$ is analytic in a neighborhood of $I = 0$ and

$$\mathrm{d}_0\omega^\star = -2\mathrm{Id}_{\ell^{p/2}_C}.$$

It follows from the inverse function theorem that $\omega^\star$ is a local diffeomorphism near $I = 0$.

(ii): We show that for any $I \in \mathcal{V}^{p/2}$ the map $\Lambda_I = \mathrm{d}_I\omega^\star + 2\mathrm{Id}_{\ell^{p/2}_{\mathbb{C}}}$ is a compact operator on $\ell^{p/2}_{\mathbb{C}}$. Since $\omega^\star_n + 2I_n = \ell^{p/3}_n + \ell^{1+}_n$, there exists $1 < r < p/2$ so that $\omega^\star + 2\,\mathrm{id}\colon \mathcal{V}^{p/2} \to \ell^r_{\mathbb{C}}$ is analytic. By Cauchy's estimate it follows that $\Lambda_I\colon \ell^{p/2}_{\mathbb{C}} \to \ell^r_{\mathbb{C}}$ is bounded and hence compact by Pitt's Theorem – see [17] for a short proof.

(iii): As an immediate consequence of item (i), $\omega^\star$ is a Fredholm map of index zero everywhere on $\mathcal{V}^{p/2}$, $p > 2$.

(iv): Since $\omega^\star\colon \mathcal{V}^{p/2} \to \ell^{p/2}_{\mathbb{C}}$, $p > 2$, is real analytic, a diffeomorphism at the origin, and its differential is a compact perturbation of $-2\mathrm{Id}_{\ell^{p/2}_{\mathbb{C}}}$, it follows from Proposition I.4 that $\omega^\star$ is a local diffeomorphism generically on $\mathcal{V}^{p/2}$. $\blacksquare$

21. NLS Hamiltonian

In this section, we derive, inspired by the work of Korotyaev [45], a novel formula for the Hamiltonian

$$\mathcal{H}^\star = \mathcal{H} - 2\mathcal{H}_1^2 - \sum_{m\in\mathbb{Z}} (2n\pi)^2 I_n,$$

which allows us to extend the function $\mathcal{H}^\star$, a priori real analytic on H^1_r, real analytically to $\mathcal{W}^4$.

For convenience we introduce for any integers $n \in \mathbb{Z}$ and $m \geqslant 0$ the moments

$$\mathcal{R}^{(m)}_n = -\frac{1}{\pi}\int_{\Gamma_n} F^m_n(\lambda)\,\mathrm{d}\lambda,$$

and collect some basic facts about them.

Lemma 21.1 *For any* $1 < p < \infty$

(i) $\mathcal{R}_n^{(m)}$ *is analytic on* $\mathcal{W}_p$ *for all* $n \in \mathbb{Z}$, $m \geqslant 0$,

(ii) $\mathcal{R}_n^{(2m)} = 0$ *for all* $n \in \mathbb{Z}$, $m \geqslant 0$.

(iii) $\mathcal{R}_n^{(m)} = O(\gamma_n^{m+1})$ *locally uniformly on* $\mathcal{W}^p$ *and uniformly as* $|n| \to \infty$. *In particular,* $\mathcal{R}_n^{(m)}$ *vanishes if* γ_n *vanishes.*

(iv) $\mathcal{R}_n^{(m)} \geqslant 0$ *on* FL_r^p. *Indeed,* $\mathcal{R}_n^{(2m+1)}$ *vanishes if and only if* γ_n *vanishes.*

(v) $\mathcal{R}_n^{(1)} = I_n$ *for all* $n \in \mathbb{Z}$. $\rtimes$

Proof. The analyticity claimed in item (i) follows immediately from the analyticity of F_n established in Lemma 19.1 on $(\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}} \overline{U_m}) \times V_\varphi$ for any $\varphi \in \mathcal{W}^p$. By the same lemma, every even power of F_n is analytic on U_n, which proves item (ii), and $\sup_{\lambda\in G_n^+\cup G_n^-} |F_n(\lambda)| = O(\gamma_n)$ which proves item (iii). Moreover, if φ is of real type, then $F_n(\lambda)\big|_{G_n^\pm} = \pm\cosh^{-1}((-1)^n\Delta(\lambda)/2)$, which proves (iv). Finally, integrating by parts in the definition of the actions (2.16), and using the fact that $F(\lambda) = F_n(\lambda) - \mathrm{i}n\pi$, yields

$$I_n = \frac{1}{\pi}\int_{\Gamma_n} \frac{\lambda\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}}\,\mathrm{d}\lambda = -\frac{1}{\pi}\int_{\Gamma_n} F(\lambda)\,\mathrm{d}\lambda = -\frac{1}{\pi}\int_{\Gamma_n} F_n(\lambda)\,\mathrm{d}\lambda. \qquad \blacksquare$$

We are now in a position to derive the novel formula for the Hamiltonian $\mathcal{H}^\star$.

Lemma 21.2 *For any finite gap potential of real type*

$$\mathcal{H} - 2\mathcal{H}_1^2 - \sum_{n\in\mathbb{Z}} (2n\pi)^2 I_n = \frac{4}{3}\sum_{n\in\mathbb{Z}} \mathcal{R}_n^{(3)}. \quad \rtimes \tag{4.13}$$

Proof. Suppose φ is a finite gap potential of real type, then $A = \{n \in \mathbb{Z} : \gamma_n \neq 0\}$ is finite and by Corollary 19.3 $F^3(\lambda)$ is analytic on $\mathbb{C} \setminus \bigcup_{n\in A} G_n$ with residue

$$\frac{1}{2\pi\mathrm{i}}\int_{C_r} F^3(\lambda) = \frac{3}{8\mathrm{i}}(\mathcal{H} - 2\mathcal{H}_1^2),$$

where C_r denotes a counter clockwise oriented circle around the origin enclosing all open gaps. On the other hand, by expanding $F^3(\lambda)$ into

$$F^3(\lambda) = (F_n(\lambda) - \mathrm{i}n\pi)^3 = F_n^3(\lambda) - 3\mathrm{i}F_n^2(\lambda)(n\pi) - 3F_n(\lambda)(n\pi)^2 + (n\pi)^3,$$

one obtains by contour deformation and the previous lemma

$$\frac{1}{\pi}\int_{C_r} F^3\,\mathrm{d}\lambda = \sum_{n\in A}\frac{1}{\pi}\int_{\Gamma_n} F^3\,\mathrm{d}\lambda = \sum_{n\in A}\Big(3(n\pi)^2\mathcal{R}_n^{(1)} - \mathcal{R}_n^{(3)}\Big).$$

Combining both identities for the residue of $F^3(\lambda)$ yields

$$\mathcal{H} - 2\mathcal{H}_1^2 = \sum_{n\in A}(2n\pi)^2 I_n - \frac{4}{3}\sum_{n\in A}\mathcal{R}_n^{(3)}. \qquad \blacksquare$$

Proposition 21.3 *The series $-\frac{4}{3}\sum_{n\in\mathbb{Z}}\mathcal{R}_n^{(3)}$ converges absolutely and locally uniformly on $\mathcal{W}^4$ to the analytic function $\mathcal{H}^\star$,*

$$\mathcal{H}^\star = \mathcal{H} - 2\mathcal{H}_1^2 - \sum_{n\in\mathbb{Z}}(2n\pi)^2 I_n = -\frac{4}{3}\sum_{n\in\mathbb{Z}}\mathcal{R}_n^{(3)}.$$

On FL_r^4 the right hand side is nonpositive and vanishes if and only if $\varphi = 0$. ⋊

Proof. By Lemma 21.1 each $\mathcal{R}_n^{(3)}$, $n \in \mathbb{Z}$, is analytic on $\mathcal{W}^4$ and we have the locally uniform estimate $\mathcal{R}_n^{(3)} = O(\gamma_n^4) = \ell_n^1$. Consequently, the series $-\frac{4}{3}\sum_{n\in\mathbb{Z}}\mathcal{R}_n^{(3)}$ converges absolutely and locally uniformly to an analytic function $\mathcal{H}^\star$ on $\mathcal{W}^4$. When restricted to FL_r^4, every functional $\mathcal{R}_n^{(3)}$ is nonnegative and vanishes if and only if $\gamma_n = 0$. Thus, $\mathcal{H}^\star$ is nonnegative on FL_r^4 and vanishes if and only if $\varphi = 0$. ∎

Proof of Theorem 18.3. (i): This is the content of Proposition 21.3. (ii): Using item (i) one shows, by the same arguments as in the proof of Theorem 20.3 in [23], that the Hamiltonian $\mathcal{H}^\star$, when viewed as a function of the actions, defines a real analytic function $H^\star\colon \mathcal{V}^2 \to \mathbb{C}$. Moreover, $H^\star$ is nonnegative on $\ell_+^2 \cap \mathcal{V}^2$, and vanishes on $\ell_+^2 \cap \mathcal{V}^2$ if and only if $I = 0$. Recall that $\omega_n^\star = \partial_{I_n} H^\star$, hence $\mathrm{d}_I^2 H^\star = \mathrm{d}_I\omega^\star$ for all $I \in \mathcal{V}^2$. Therefore, in view of the proof of Corollary 18.2, at $I = 0$

$$\mathrm{d}_0^2 H^\star = \mathrm{d}_0\omega^\star = -2\mathrm{Id}_{\ell_{\mathbb{C}}^2}.$$

The strict concavity of $H^\star$ in a neighborhood of $I = 0$ thus follows by continuity. ⋊

22. NLS wellposedness

The frequencies of the renormalized NLS equation are given by

$$\omega_n^r = \omega_n - 4H_1 = (2n\pi)^2 + \omega_n^\star.$$

The frequencies $\omega_n^\star$ extend to real analytic functions on FL_r^p for any $1 < p < \infty$ by Theorem 20.5 from the previous section. Consequently, the frequencies ω_n^r give rise to a flow in Birkhoff coordinates on ℓ_r^p, $1 < p < \infty$, defined globally in time for all initial values $\Omega_p(\varphi)$, $\varphi \in FL_r^p$.

It turns out to be convenient to consider the complex Birkhoff coordinates

$$z_n = \frac{x_n - \mathrm{i}y_n}{\sqrt{2}}, \quad w_n = \frac{x_n + \mathrm{i}y_n}{\sqrt{2}}, \quad n \in \mathbb{Z},$$

where for real type states we have $w_n = \overline{z_n}$. By a slight abuse of notation, we also denote the space of complex Birkhoff coordinates (z, w) by ℓ_c^p and the subspace of real type states by ℓ_r^p. Moreover, also the complex Birkhoff map $\varphi \mapsto (z, w)$ is denoted by the symbol Ω_p. This notation is local to this section and should not lead to any confusion.

The flow map of the frequencies ω_n^r in complex Birkhoff coordinates is then given by $S_\Omega^r\colon (t, (z, w)) \mapsto (Z_n^r(t, (z, w)), W_n^r(t, (z, w)))$ with coordinate functions

$$Z_n^r(t, (z, w)) = \mathrm{e}^{\mathrm{i}t\omega_n^r(z,w)} z_n, \qquad W_n^r(t, (z, w)) = \mathrm{e}^{-\mathrm{i}t\omega_n^r(z,w)} w_n, \tag{4.14}$$

which are real analytic functions on $\mathbb{R} \times \Omega_p(FL_r^p)$.

The $(\mathrm{dNLS})_r$ solution map on FL_r^p is then given by

$$S^r(t, \varphi) = \Omega_p^{-1}(S_\Omega^r(t, \Omega_p(\varphi))), \tag{4.15}$$

and well defined for all t such that $S^r_\Omega(t,\Omega_p(\varphi))$ is in the image of Ω_p.

Similarly, the dNLS frequencies

$$\omega_n = (2n\pi)^2 + 4S_1 + \omega_n^\star$$

give rise to a flow S_Ω on ℓ^p_r for all $1 < p \leqslant 2$. The corresponding dNLS solution map on FL^p_r is given by

$$S(t,\varphi) = \Omega_p^{-1}(S_\Omega(t,\Omega_p(\varphi))), \tag{4.16}$$

and well defined for all $t \in \mathbb{R}$ since $\Omega_p\colon FL^p_r \to \ell^p_r$ is onto if $1 < p \leqslant 2$.

We first consider properties of the map S_Ω corresponding to the ones of S_Ω claimed in Theorem 18.5.

Theorem 22.1 *(i) Suppose* $1 < p < \infty$. *For any* $(z,w) \in \Omega_p(FL^p_r) \subset \ell^p_r$, *the curve*

$$\mathbb{R} \to \ell^p_r, \qquad t \mapsto S^r_\Omega(t,(z,w)),$$

is continuous. Moreover, for any $T > 0$ *the map* $S^r_\Omega\colon \Omega_p(FL^p_r) \to C([-T,T],\ell^p_r)$ *is real analytic. If* $1 < p \leqslant 2$, *this map has the group property and* $S^r_\Omega(t,\cdot)\colon \ell^p_r \to \ell^p_r$ *for any* $t \in \mathbb{R}$ *is a diffeomorphism.*

(ii) Suppose $1 < p \leqslant 2$. *For any* $(z,w) \in \ell^p_r$, *the curve* $\mathbb{R} \to \ell^p_r$, $t \mapsto S_\Omega(t,(z,w))$ *is continuous. Moreover, for any* $T > 0$ *the map* $S_\Omega\colon \ell^p_r \to C([-T,T],\ell^p_r)$ *is real analytic, has the group property, and* $S_\Omega(t,\cdot)\colon \ell^p_r \to \ell^p_r$ *is a diffeomorphism for any* $t \in \mathbb{R}$.

(iii) For each $2 < p < \infty$, $n \in \mathbb{Z}$, *and* $T > 0$, *the NLS coordinate functions*

$$Z_n, W_n\colon \Omega_p(FL^p_r) \subset \ell^p_r \to C([-T,T],\mathbb{C})$$

cannot be extended continuously to points $(z,w) \in \Omega_p(FL^p_r) \setminus \ell^2_r$ *with* $z_n \neq 0$ *and* $w_n \neq 0$, *respectively.* ⋊

Proof. (i): The proof is analogous to the one for KdV found in [38, Section 4]. Clearly, in view of (4.14) each coordinate function is defined globally in time, and we have for any $t,s \in \mathbb{R}$ and any $n \in \mathbb{Z}$

$$\begin{aligned} |Z^r_n(t,(z,w)) - Z^r_n(s,(z,w))| &\leqslant |\mathrm{e}^{\mathrm{i}t\omega^r_n(z,w)} - \mathrm{e}^{\mathrm{i}t\omega^r_n(z,w)}||z_n| \\ &\leqslant |\mathrm{e}^{\mathrm{i}(t-s)\omega^r_n(z,w)} - 1||z_n|, \end{aligned}$$

and similarly for $|W^r_n(t,(z,w)) - W^r_n(s,(z,w))|$. Consequently, for any $N \geqslant 1$

$$\begin{aligned} \|S^r_\Omega(t,(z,w)) - S^r_\Omega(s,(z,w))\|_p^p &\leqslant \sum_{|n|\leqslant N} |\mathrm{e}^{\mathrm{i}(t-s)\omega^r_n(z,w)} - 1|^p(|z_n|^p + |w_n|^p) \\ &\quad + 2^p \sum_{|n|>N} (|z_n|^p + |w_n|^p). \end{aligned}$$

Given $\varepsilon > 0$, we first choose $N \geqslant 1$ so that $2^p\sum_{|n|>N}(|z_n|^p + |w_n|^p) \leqslant \varepsilon^p$, and second $\delta > 0$, so that for all $t,s \in \mathbb{R}$ with $|t-s| < \delta$ we have

$$\sup_{|n|\leqslant N} |\mathrm{e}^{\mathrm{i}(t-s)\omega^r_n(z,w)} - 1|\|(z,w)\|_p \leqslant \varepsilon.$$

Then $\|S^r_\Omega(t,(z,w)) - S^r_\Omega(s,(z,w))\|_p \leqslant 2\varepsilon$, which shows that the curve $t \mapsto S^r_\Omega(t,(z,w))$ is a continuous curve in ℓ^p_r and the group property follows immediately.

Next, we show that the map $S^r_\Omega\colon \Omega_p(FL^p_r) \subset \ell^p_r \to C([-T,T],\ell^p_r)$ is real analytic for any $T>0$. Indeed, each coordinate function is a real analytic map $Z^r_n\colon \Omega_p(FL^p_r) \to C([-T,T],\mathbb{R})$ for any $T>0$. Moreover, for any $(z,w)\in\Omega_p(FL^p_r)$ there exists a neighborhood V in ℓ^p_c so that $\omega^r_n = (2n\pi)^2 + \ell^{p/2}_n + \ell^{1+}_n$ uniformly on V, hence

$$Z^r_n(t,(z,w)) = \mathrm{e}^{\mathrm{i}t\omega^r_n(z,w)} z_n = \mathrm{e}^{\mathrm{i}t(2n\pi)^2}\left(1+\ell^{p/2}_n+\ell^{1+}_n\right) z_n = \ell^p_n$$

uniformly on $[-T,T]\times V$ for any $T>0$. Consequently, $S^r_\Omega\colon V \to C([-T,T],FL^p_c)$ is analytic.

(ii): the proof is the same as the one for item (i) with the sole difference that the dNLS frequencies are defined on all of ℓ^p_r provided $1<p\leqslant 2$.

(ii): Take any initial datum $(z,w)\in\Omega_p(FL^p_r)\setminus \ell^2_r$. Denote by $(z^{(m)},w^{(m)})$ any sequence in ℓ^2_r which converges to (z,w) in ℓ^p_r. Each functional ω^r_n is real analytic on $\Omega_p(FL^p_r)$ for any $p>2$ whereas $S_1(z,w) = \sum_{m\in\mathbb{Z}} z_m w_m = \frac{1}{2}\sum_{m\in\mathbb{Z}}(|z_m|^2+|w_m|^2)$ is infinite on ℓ^p_r. We conclude that $\omega_n(z^{(m)},w^{(m)}) \to +\infty$. Choose $n\geqslant 1$ so that $\varepsilon = |z_n|>0$ and fix $T>0$. After possibly passing to a subsequence of $(z^{(m)},w^{(m)})$, we can choose a sequence of real numbers $-T\leqslant t_m\leqslant T$ so that

$$|\mathrm{e}^{\mathrm{i}(\omega_n(z^{(m+1)},w^{(m+1)})-\omega_n(z^{(m)},w^{(m)}))t_m} - 1| \geqslant 1, \qquad m\geqslant 1.$$

Consequently,

$$\begin{aligned} |Z_n(t_m,(z^{(m+1)},w^{(m+1)})) - Z_n(t_m,(z^{(m)},w^{(m)}))| &\geqslant |z^{(m)}_n| - \|z^{(m+1)}-z^{(m)}\|_p \\ &\geqslant \varepsilon/2 \end{aligned}$$

for all m sufficiently large. ∎

Theorem 18.5 is implied by the following result.

Corollary 22.2 *(i) For any $1<p\leqslant 2$, dNLS and (dNLS)$_r$ are globally in time C^ω-wellposed in FL^p_r.*

(ii) For any $2<p<\infty$, (dNLS)$_r$ is locally in time C^ω-wellposed in FL^p_r and there exists a neighborhood U^p of the origin on which (dNLS)$_r$ is globally in time C^ω-wellposed.

(iii) For any $2<p<\infty$, dNLS is illposed in FL^p_r in the sense that for any $T>0$ and any $q\geqslant p$ the solution map cannot be extended to a map $S\colon FL^p_r \to C([0,T],FL^q_r)$ which is continuous at any point of $FL^p_r\setminus L^2_r$. ⋊

Proof. (i) For $1<p\leqslant 2$ the Birkhoff map $\Omega_p\colon FL^p_r \to \ell^p_r$ is a bi real analytic diffeomorphism, hence the Solution maps (4.15) and (4.16) extend globally in view of Theorem 22.1.

(ii) Suppose $p>2$. For any point $\varphi\in FL^p_r$ there exists a neighborhood U of φ in FL^p_r and a neighborhood V of $\Omega_p(\varphi)$ in ℓ^p_r so that $\Omega_p|_U\colon U\to V$ is a diffeomorphism. Since $S^r_\Omega\colon \mathbb{R}\times U\to\ell^p_r$ is continuous by Theorem 22.1, there exists $T>0$ and a neighborhood $V'\subset V$ of $\Omega_p(\varphi)$ so that $S^r_\Omega([-T,T],V')\subset V$. Let $U' = \Omega_p^{-1}(V')$, then

$$S^r(\cdot,\psi) = \Omega_p^{-1}(S^r_\Omega(\cdot,\Omega_p(\psi))), \tag{4.17}$$

is well defined on $[-T,T]$ for any $\psi \in U'$ and defines a real analytic map $S^r\colon U' \to C([-T,T],FL^p_r)$. This proves the local-in-time C^ω-wellposedness in FL^p_r.

Suppose $V = B_r(0)$ is a ball of radius $r > 0$ centered at the origin of FL^p_r. In view of (4.14), we have $\|S^r_\Omega(t,(z,w))\|_p = \|(z,w)\|_p$ for all $t \in \mathbb{R}$, hence

$$S^r_\Omega(t,V) \subset V, \qquad t \in \mathbb{R}.$$

Therefore, (4.17) is well defined on $\mathbb{R}$ for any $\psi \in U = \Omega_p^{-1}(V)$. In consequence, the map $S^r\colon U' \to C([-T,T],FL^p_r)$ is real analytic for any $T > 0$. This proves the global-in-time C^ω-wellposedness in U.

(iii) Suppose there exists $\varphi \in FL^p_r \setminus L^2_r$, $p > 2$, so that $S\colon FL^p_r \mapsto C([0,T],FL^q_r)$ is continuous at φ for any $T > 0$ and any $p \leqslant q < \infty$. Then also

$$S_\Omega = \Omega_q \circ S \circ \Omega_p^{-1}\colon \Omega_p(FL^p_r) \subset \ell^p_r \to C([0,T],\ell^q_r)$$

is continuous at $\Omega_p(\varphi)$ which contradicts Theorem 22.1 (iii). ∎

Remark 22.3. A continuous curve $\gamma\colon [-T,T] \to FL^p_r$, $1 < p < \infty$, is a solution of (dNLS)$_r$ if and only if its coordinate functions $(z_n(t),w_n(t))_{n\in\mathbb{Z}} = \Omega \circ \gamma(t)$ satisfy (dNLS)$_r$ in Birkhoff coordinates, that is

$$z_n(t) = \mathrm{e}^{\mathrm{i}t\omega^r_n(z(0),w(0))}z_n(0), \qquad w_n(t) = \mathrm{e}^{-\mathrm{i}t\omega^r_n(z(0),w(0))}w_n(0).$$

Consequently, the uniqueness of the solutions constructed in this chapter holds in the whole class $C([-T,T],FL^p_r)$, slightly improving on the results of Grünrock & Herr [25] where the uniqueness was obtained in certain restriction norm spaces of Bourgain spaces embedding continuously into $C([-T,T],FL^p_r)$. ⊸

Chapter 5

Two sided estimates for the Birkhoff map in Sobolev spaces

23. Overview

In the preceding chapters we investigated the Birkhoff coordinates for the dNLS equation in spaces of low regularity. However, these coordinates are also well suited to investigate the dNLS equation in spaces of high regularity. Recall that the *Birkhoff map* $\varphi \mapsto (x_n, y_n)_{n\in\mathbb{Z}}$ constructed in the prequel, when restricted to the Sobolev spaces H_r^m, $m \geqslant 0$, is a bi-analytic, canonical diffeomorphism $\Omega\colon H_r^m \to h_r^m$, and on h_r^1 the transformed NLS Hamiltonian is a real analytic function of the actions $I_n = (x_n^2 + y_n^2)/2$ alone – see also [23]. The transformed dNLS flows then takes the particularly simple form

$$\dot{x}_n = -\omega_n y_n, \quad \dot{y}_n = \omega_n x_n, \qquad \omega_n \coloneqq \partial_{I_n} H.$$

One may thus think of Ω as a nonlinear Fourier transform for the dNLS equation. Indeed, the derivative $\mathrm{d}_0\Omega$ of Ω at the origin *is* the Fourier transform, and on L_r^2, as for the Fourier transform,

$$\|\Omega(\varphi)\|_{\ell_r^2} = \|\varphi\|_{L_r^2},$$

which we also refer to as Parseval's identity – cf. e.g. [55, 23]. The main result of this chapter says that also for higher order Sobolev norms there exist analogs of Parseval's identity for the nonlinear map Ω. To give a precise statement, let $\sum_{n\in\mathbb{Z}}(\varphi_{2n}^- \mathrm{e}^{-\mathrm{i}2n\pi x}, \varphi_{2n}^+ \mathrm{e}^{\mathrm{i}2n\pi x})$ denote the Fourier series of $\varphi = (\varphi_-, \varphi_+) \in H_c^s$, $s \geqslant 0$, and endow the Sobolev space H_c^s with the norm $\|\varphi\|_s$ given by

$$\|\varphi\|_{H^s}^2 \coloneqq \sum_{n\in\mathbb{Z}} \langle 2n\pi\rangle^{2s} (|\varphi_{2n}^-|^2 + |\varphi_{2n}^+|^2), \qquad \langle x\rangle \coloneqq 1 + |x|.$$

Furthermore, the norm on the model space

$$h_r^s \coloneqq \{(x,y) = (x_n, y_n)_{n\in\mathbb{Z}} : \|(x,y)\|_s \coloneqq (\|x\|_s^2 + \|y\|_s^2)^{1/2} < \infty\},$$

is given by

$$\|x\|_{h^s}^2 \coloneqq \sum_{n\in\mathbb{Z}} \langle 2n\pi\rangle^{2s} |x_n|^2.$$

Theorem 23.1 *For any integer $m \geqslant 1$ there exist absolute constants $c_m, d_m > 0$ such that the restriction of Ω to H_r^m satisfies the two sided estimates*

$$\text{(i)} \quad \|\Omega(\varphi)\|_{h_r^m} \leqslant c_m \left(\|\varphi\|_{H_r^m} + (1 + \|\varphi\|_{H_r^1})^{2m} \|\varphi\|_{H_r^0}\right),$$

and

$$\text{(ii)} \quad \|\varphi\|_{H_r^m} \leqslant d_m \left(\|\Omega(\varphi)\|_{h_r^m} + (1 + \|\Omega(\varphi)\|_{h_r^1})^{4m-3} \|\Omega(\varphi)\|_{h_r^0}\right). \quad \rtimes$$

The main feature of Theorem 23.1 is that the constants c_m and b_m can be chosen independently of φ.

Note that the estimate (i) is linear in the highest Sobolev norm $\|\varphi\|_{H_r^m}$ for $m \geqslant 2$, and that the estimate (ii) is linear in the highest weighted h_r^m-norm $\|\Omega(\varphi)\|_{h_r^m}$ of $\Omega(\varphi)$. Hence Theorem 23.1 shows that the 1-smoothing property of the Birkhoff map Ω established in [39] holds in a uniform fashion.

The proof of Theorem 23.1 relies on estimates of the action variables $I = (I_n)_{n\in\mathbb{Z}}$ of φ, where $I_n = (x_n^2 + y_n^2)/2$, $n \in \mathbb{Z}$. The decay properties of the actions I_n are known to be closely related to the regularity properties of φ – c.f. [34, 18, 37]. We need to quantify this relationship by providing two sided estimates of the Sobolev norms of φ in terms of weighted ℓ^1-norms of $I(\varphi)$.

Theorem 23.2 *For any integer $m \geqslant 1$, there exist absolute constants c_m and d_m, such that for all $\varphi \in H_r^m$*

$$\text{(i)} \quad \|I(\varphi)\|_{\ell^{2m,1}} \leqslant c_m^2 \left(\|\varphi\|_{H_r^m}^2 + (1 + \|\varphi\|_{H_r^1})^{4m} \|\varphi\|_{H_r^0}^2\right),$$

$$\text{(ii)} \quad \|\varphi\|_{H_r^m}^2 \leqslant d_m^2 \left(\|I(\varphi)\|_{\ell^{2m,1}} + (1 + \|I(\varphi)\|_{\ell^{2,1}})^{4m-3} \|I(\varphi)\|_{\ell^1}\right). \quad \rtimes$$

Remark 23.3. The same constants c_m, d_m of Theorem 23.1 can be used in Theorem 23.2. ⊸

It turns out that versions of the estimate (i) of Theorems 23.1 & 23.2 hold for a larger family of spaces known as *weighted Sobolev spaces* – see Section 6 and [28, 29] for an introduction. A *normalized, submultiplicative*, and *monotone weight* is a symmetric function $w\colon \mathbb{Z} \to \mathbb{R}$ with

$$w_n \geqslant 1, \qquad w_n = w_{-n}, \qquad w_{n+m} \leqslant w_n w_m, \qquad w_{|n|} \leqslant w_{|n|+1}, \tag{5.1}$$

for all $n, m \in \mathbb{Z}$. The class of all such weights is denoted by $\mathcal{M}$ and H_c^w denotes the space of L_c^2 functions φ with finite w-norm

$$\|\varphi\|_w := \left(\sum_{m\in\mathbb{Z}} w_m^2 (|\varphi_m^-|^2 + |\varphi_m^+|^2)\right)^{1/2}.$$

The space H_r^w is defined analogously. Further, we denote by h_r^w the subspace of ℓ_r^2 with elements (x, y) so that $\|x\|_w^2 + \|y\|_w^2 < \infty$,

$$\|x\|_w^2 := \sum_{n\in\mathbb{Z}} w_{2n}^2 |x_n|^2.$$

For any $s \geqslant 0$, the *Sobolev weight* $\langle n\pi\rangle^s := (1 + |n\pi|)^s$ gives rise to the usual Sobolev space H_c^s. For $s \geqslant 0$ and $a > 0$, the *Abel weight* $\langle n\pi\rangle^s e^{a|n|}$ gives rise to the space $H_c^{s,a}$ of L_c^2-functions, which can be analytically extended to the open strip $\{z : |\Im z| < a/2\pi\}$ of the complex plane with traces in H_c^s on the boundary lines. In between are, among others, the *Gevrey weights*

$$\langle n\rangle^s \mathrm{e}^{a|n|^\sigma}, \qquad 0 < \sigma < 1, \quad s \geqslant 0, \quad a > 0,$$

which give rise to the Gevrey spaces $H_c^{s,a,\sigma}$, as well as weights of the form

$$\langle n\rangle^s \exp\left(\frac{a|n|}{1+\log^\sigma\langle n\rangle}\right), \qquad 0 < \sigma < 1, \quad s \geqslant 0, \quad a > 0,$$

that are lighter than Abel weights but heavier than Gevrey weights.

To avoid certain technicalities in our estimates, we restrict ourselves to weights incorporating a factor which grows at a linear rate. We thus introduce the subclass

$$\mathcal{M}^1 \coloneqq \{w \in \mathcal{M} : w_n = \langle n\rangle v_n \text{ for all } n \in \mathbb{Z} \text{ with some } v \in \mathcal{M}\}.$$

Finally, we assume all weights $w \in \mathcal{M}$ to be piecewise linearly extended to functions on the real line $w\colon \mathbb{R} \to \mathbb{R}_{>0}$, $t \mapsto w[t]$.

Theorem 23.4 *For any weight $w \in \mathcal{M}^1$ there exists a complex neighborhood $\mathcal{W}^w$ of H_r^w within H_c^w and a constant c_w, such that*

$$\sum_{n\in\mathbb{Z}} w_{2n}^2 |I_n| \leqslant c_w^2 w[16\|\varphi\|_w^2]^2 \|\varphi\|_w^2.$$

Moreover, the restriction of the Birkhoff map Ω to H_r^w takes values in h_r^w and satisfies

$$\|\Omega(\varphi)\|_w \leqslant c_w w[16\|\varphi\|_w^2]\|\varphi\|_w. \quad \rtimes$$

In this more general set up the bounds of Theorem 23.4 are of the same type as the weight, and are valid for all submultiplicative weights including those growing exponentially fast. Note that in view of (5.1) one has $w_m \leqslant w_0^{|m|}$, hence a submultiplicative weight cannot grow faster then exponentially. The following version of Theorem 23.4 for Sobolev spaces of real exponent complements the results of Theorems 23.2-23.4.

Corollary 23.5 *For any real $s \geqslant 1$ there exist a complex neighborhood $\mathcal{W}^s$ of H_r^s and a constant c_s such that*

$$\|I(\varphi)\|_{\ell_{\mathbb{C}}^{2s,1}} \leqslant c_s^2(1+\|\varphi\|_{H_c^s})^{4s}\|\varphi\|_{H_c^s}^2.$$

Moreover, the restriction of the Birkhoff map Ω to H_r^s takes values in h_r^s and satisfies

$$\|\Omega(\varphi)\|_{h_r^s} \leqslant c_s(1+\|\varphi\|_{H_r^s})^{2s}\|\varphi\|_{H_r^s}. \quad \rtimes$$

Outline of this chapter. We consider the action variables $J_{n,k}$ on levels $k \geqslant 1$ introduced by McKean & Vaninsky [55]. For $k = 1$ they coincide with the ordinary actions I_n, while their asymptotic behavior on odd levels $k = 2m+1$, $m \geqslant 0$, turns out to be

$$J_{n,2m+1} \sim (\lambda_n^\pm)^{2m} I_n \sim (n\pi)^{2m} I_n, \qquad |n| \to \infty.$$

Moreover, they satisfy the trace formula

$$\sum_{n\geqslant 1} J_{n,2m+1} = \frac{(-1)^{m+1}}{4^m}\mathcal{H}_{2m+1}, \qquad m \geqslant 1,$$

where $\mathcal{H}_{2m+1}$ denotes the $2m+1$th Hamiltonian in the NLS-hierarchy. On H_r^m with $\varphi = (\psi, \overline{\psi})$,

$$\mathcal{H}_1 = \int_{\mathbb{T}} |\psi|^2 \, \mathrm{d}x, \quad \mathcal{H} = \mathcal{H}_3 = \int_{\mathbb{T}} (|\psi'|^2 + |\psi|^4) \, \mathrm{d}x, \quad \dots,$$

and more generally for any $m \geqslant 1$,

$$\mathcal{H}_{2m+1} = \int_{\mathbb{T}} \left(|\psi^{(m)}|^2 + p_m(\psi, \dots, \psi^{(m-1)}) \right) \mathrm{d}x,$$

where p_m is a polynomial expression in ψ and its first $m-1$ derivatives. Viewing $\mathcal{H}_{2m+1}$ as a lower order perturbation of the H_r^m-norm we obtain at first order

$$\sum_{n \in \mathbb{Z}} (n\pi)^{2m} I_n \sim \sum_{n \in \mathbb{Z}} J_{n,2m+1} \sim \|\varphi\|_{H_r^m}^2.$$

The essential ingredient to make this formal statement explicit is a sufficiently accurate localization of the periodic eigenvalues $\lambda_n^{\pm}$ of φ in H_c^1 which is valid for all $n \in \mathbb{Z}$ with $|n|$ above a certain threshold depending only on $\|\varphi\|_{H^1}$. Above this threshold we can directly compare the weighted action norms and the polynomial expressions in φ as described above, while the finitely many terms below the threshold can be regard as a H^1-error term. Thereof we obtain Theorem 23.2, which directly implies Theorem 23.1. Note that our method of proof completely avoids the use of auxiliary spectral quantities such as *spectral heights* or *conformal mapping theory*.

To prove Theorem 23.4, we take a slightly different approach by quantitatively estimating the action variables in terms of the gap lengths of the associated Zakharov-Shabat operator. For the latter we obtain estimates in any weighted norm, which allows us to obtain Theorem 23.4.

24. Actions on level k and trace formulae

In this section we recall from [23, 55] the definition of the actions on level $k \geqslant 1$ as well as the trace formulae. In Section 11, the action variables of $\varphi \in \mathcal{W} \equiv \mathcal{W}^2$ have been defined by

$$I_n = \frac{1}{\pi} \int_{\Gamma_n} \frac{\lambda \Delta^{\bullet}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} \, \mathrm{d}\lambda,$$

where Γ_n denotes any counter clockwise oriented circuit which is circling sufficiently close around G_n. The *nth action on level* $k \geqslant 1$ is now given by

$$J_{n,k} \coloneqq \frac{1}{k\pi} \int_{\Gamma_n} \frac{\lambda^k \Delta^{\bullet}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} \, \mathrm{d}\lambda.$$

Clearly, on level one we get back the ordinary actions. For $k \geqslant 2$, by the same arguments as in Section 11, each action variable $J_{n,k}$ is analytic on $\mathcal{W}$ and vanishes if and only if γ_n is zero – see also [23, Theorem 13.5]. Moreover, we can integrate by parts in the definition of the action variable to obtain the representation

$$J_{n,k} = -\frac{1}{\pi} \int_{\Gamma_n} \lambda^{k-1} F(\lambda) \, \mathrm{d}\lambda, \tag{5.2}$$

where $F(\lambda)$ denotes the primitive of $\frac{\Delta^{\bullet}(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}$ introduced in Section 19.

The actions on level one are well known to satisfy the trace formula,

$$\sum_{n \in \mathbb{Z}} I_n(\varphi) = \mathcal{H}_1(\varphi) = \frac{1}{2} \|\varphi\|_{L^2}^2 = \frac{1}{2} \int_{\mathbb{T}} (|\varphi_-|^2 + |\varphi_+|^2) \, \mathrm{d}x. \tag{5.3}$$

Similar trace formulae have been derived by McKean & Vaninsky [55] expressing the actions on any level $k \geqslant 1$ in terms of Hamiltonians of the *NLS-hierarchy*. The first three Hamiltonians of this hierarchy[1] are

$$\begin{aligned} \mathcal{H}_1(\varphi) &= \int_{\mathbb{T}} \varphi_- \varphi_+ \,\mathrm{d}x, \\ \mathcal{H}_2(\varphi) &= \frac{\mathrm{i}}{2} \int_{\mathbb{T}} (\varphi_-' \varphi_+ - \varphi_- \varphi_+') \,\mathrm{d}x, \\ \mathcal{H}_3(\varphi) &= \int_{\mathbb{T}} (\varphi_-' \varphi_+' + \varphi_+^2 \varphi_-^2) \,\mathrm{d}x, \end{aligned}$$

and in general, for any sufficiently regular $\varphi \in L_c^2$,

$$\mathcal{H}_{k+1}(\varphi) = (-\mathrm{i})^k \int_{\mathbb{T}} (\varphi_- \varphi_+^{(k)} + q_k(\varphi, \dots, \varphi^{(k-1)})) \,\mathrm{d}x, \qquad k \geqslant 1,$$

with q_k being a canonically determined polynomial in φ and its first $k-1$ derivatives – see Appendix H. The following version of the trace formula was proved in [23, Theorem 13.6].

Theorem 24.1 (Trace Formula) *For any $k \geqslant 1$ and any $\varphi \in H_c^{k-1} \cap \mathcal{W}$,*

$$\sum_{n \in \mathbb{Z}} J_{n,k}(\varphi) = \frac{1}{2^{k-1}} \mathcal{H}_k(\varphi). \quad \rtimes \tag{5.4}$$

In particular, for every sufficiently regular real type potential

$$\sum_{n \in \mathbb{Z}} J_{n,2m+1}(\varphi) = \frac{1}{4^m} \int_{\mathbb{T}} (|\varphi^{(m)}|^2 + \cdots) \,\mathrm{d}x, \qquad m \geqslant 0.$$

This identity is used in Sections 26 and 27 to estimate the actions on level $2m+1$ in terms of $\|\varphi\|_{H_r^m}$.

For φ of real type, we may shrink in formula (5.2) the contour of integration Γ_n to the interval $[\lambda_n^-, \lambda_n^+]$, and use the properties of F noted in Lemma 19.1 to the effect that

$$J_{n,k} = \frac{2}{\pi} \int_{\lambda_n^-}^{\lambda_n^+} \lambda^{k-1} f_n(\lambda) \,\mathrm{d}\lambda, \qquad f_n(\lambda) = \cosh^{-1} \frac{(-1)^n \Delta(\lambda)}{2}. \tag{5.5}$$

Thus on L_r^2 all actions are real and those on odd levels are nonnegative. Moreover, by the mean value theorem,

$$J_{n,2m+1} = \zeta_{n,m}^{2m} I_n, \tag{5.6}$$

for some $\zeta_{n,m} \in [\lambda_n^-, \lambda_n^+]$. Since $\lambda_n^\pm = n\pi + \ell_n^2$, we conclude for any $m \geqslant 0$ that $\zeta_{n,m}^{2m} \sim (n\pi)^{2m}$ as $|n| \to \infty$. To obtain a quantitative estimate of the high level actions $J_{n,2m+1}$ in terms of the actions I_n, we need a quantitative estimate of the periodic eigenvalues which will be obtained in the next section.

25. Uniform localization of the Zakharov-Shabat Spectrum

The goal for this section is to provide a sufficiently accurate localization of the spectrum of the Zakharov-Shabat operator

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix},$$

allowing to quantify the asymptotic relation $J_{n,2m+1} \sim (n\pi)^{2m} I_n$. Since one can translate the spectrum of φ without changing the L^2-norm, one can not obtain a uniform localization on bounded subsets of L_c^2. Instead, we have to impose some regularity on φ.

[1] In comparison to [23] we multiplied for $k \geqslant 2$ the Hamiltonian $\mathcal{H}_k$ with the factor $(-\mathrm{i})^{k+1}$.

Theorem 25.1 ([59]) *Suppose $\varphi \in H_c^1$, then for each $\langle n \rangle \geqslant 8\|\varphi\|_1^2$,*

$$|\lambda_n^\pm - n\pi| \leqslant \frac{\|\varphi\|_{H^1}^2}{\langle n \rangle} + \frac{\sqrt{2}\|\varphi\|_{H^1}}{\langle 2n \rangle} \leqslant \pi/5,$$

while the remaining eigenvalues are contained in the box

$$\left\{\lambda \in \mathbb{C} : |\Re\lambda| \leqslant (8\|\varphi\|_{H^1}^2 - 1/2)\pi, \quad |\Im\lambda| \leqslant \|\varphi\|_{H^1}\right\}. \quad \rtimes$$

Remark. In [53] Li & McLaughlin obtained a localization for φ in H_c^1 where the bound on the threshold of $\langle n \rangle$ is exponential in the norm of φ. With a focus on lowering the regularity assumptions on φ rather than improving the threshold for φ smooth, this result was gradually improved by several authors – see e.g. Mityagin [56] and the references therein. The key point of Theorem 25.1 consists in providing a threshold for $\langle n \rangle$ which is quadratic in the norm of φ.

The proof of Theorem 25.1 is based on the Lyapunov-Schmidt decomposition of the operator $L(\varphi)$ introduced by Kappeler & Mityagin [28] and explained in detail in Section 6 with the difference that the weights considered here contain at least a linearly growing factor, that is $\varphi \in H_c^1$. Let us briefly recall the main ideas and notations. Since for the zero potential each $n\pi$, $n \in \mathbb{Z}$, is a double eigenvalue of L with eigenfunctions $e_n^\pm$, the first step is to separate these modes from the others to simplify the analysis of the eigenvalue equation $L(\varphi)f = \lambda f$ for λ close to $n\pi$ and $|n|$ sufficiently large. To this end, we introduce the closed strips

$$\mathfrak{U}_n = \{\lambda : |\Re\lambda - n\pi| \leqslant \pi/2\},$$

and consider the splitting

$$H_c^w(\mathbb{T}_2) = \mathcal{P}_n \oplus \mathcal{Q}_n = \mathrm{sp}\{e_n^+, e_n^-\} \oplus \overline{\mathrm{sp}}\{e_k^+, e_k^- : k \neq n\},$$

where the orthogonal projections onto $\mathcal{P}_n$ and $\mathcal{Q}_n$ are denoted by P_n and Q_n, respectively. The eigenvalue equation $L(\varphi)f = \lambda f$ can then be written in the form

$$A_\lambda f = \Phi f,$$

where $A_\lambda = \lambda - L(0)$ and $\Phi = \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}$. The operator A_λ is a Fourier multiplier and hence leaves the spaces $\mathcal{P}_n$ and $\mathcal{Q}_n$ invariant. By writing

$$f = u + v = P_n f + Q_n f,$$

we can decompose the equation $A_\lambda f = \Phi f$ into the two equations

$$A_\lambda u = P_n \Phi(u + v), \qquad A_\lambda v = Q_n \Phi(u + v),$$

called the P- and the Q-equation, respectively. For all $\lambda \in \mathfrak{U}_n$ the restriction of A_λ to $\mathcal{Q}_n$ is boundedly invertible, hence , multiplying the Q-equation from the left by ΦA_λ^{-1}, gives

$$\Phi v = T_n \Phi(u + v), \qquad T_n = \Phi A_\lambda^{-1} Q_n.$$

The latter identity may be written as

$$(I - T_n)\Phi v = T_n \Phi u,$$

so that solving the Q equation amounts to inverting $(I - T_n)$. By Lemma 6.4 the operator T_n is bounded on H_c^w uniformly in $\lambda \in \mathfrak{U}_n$ for any $\varphi \in H_c^w$ with $w \in \mathcal{M}$. More to the point, for any $i \in \mathbb{Z}$ we have the following estimate in the shifted norm uniformly in $\lambda \in \mathfrak{U}_n$

$$\|T_n f\|_{w;i} \leqslant 2\|\varphi\|_w \|f\|_{w;-i}.$$

The fact that the weight w contains a factor growing linearly in n comes into play when estimating the operator norm of T_n^2. Recall from Lemma 6.5 that $\|T_n^2\|_{w;n} \leqslant 1/2$ uniformly in $\lambda \in \mathfrak{U}_n$ for all $|n| \geqslant N$ where the threshold N can be chosen locally uniformly in φ. If the weight contains a factor $\langle n\rangle$, then the threshold N can be chosen uniformly on bounded subsets of H_c^w – see also [18, 21] for weights with factors $\langle n\rangle^\delta$, $0 < \delta < 1/2$. We provide a refinement of Lemma 6.5 for the case $w \in \mathcal{M}^1$ which allows us to obtain a *quadratic* localization of the Zakharov-Shabat spectrum.

Lemma 25.2 ([59]) *If $\varphi \in H_c^w$ with $w \in \mathcal{M}^1$, then for any $n \in \mathbb{Z}$ and any $\lambda \in \mathfrak{U}_n$,*

$$\|T_n^2\|_{w;n} \leqslant \frac{4}{\langle n\rangle}\|\varphi\|_w^2. \quad \rtimes$$

Proof. As in the proof of Lemma 6.5, write $T_n^2 f = \Phi g$ with

$$g \coloneqq A_\lambda^{-1} Q_n \Phi A_\lambda^{-1} Q_n f.$$

A straightforward computation yields

$$g = \sum_{k,l\neq n}\left(\frac{\varphi_{k+l}^-}{\lambda - k\pi}\frac{f_l^+}{\lambda - l\pi}e_k^- + \frac{\varphi_{k+l}^+}{\lambda - k\pi}\frac{f_l^-}{\lambda - l\pi}e_k^+\right) = (g_-, g_+),$$

and our aim is to estimate the weighted ℓ^1-norms $\|g_+\mathrm{e}_{-n}\|_{\ell^{w,1}}$ and $\|g_-\mathrm{e}_n\|_{\ell^{w,1}}$. By assumption $w = \langle n\rangle \cdot v$ with some submultiplicative weight v, hence

$$w_{k-n} \leqslant \frac{\langle k-n\rangle}{\langle k+l\rangle\langle l+n\rangle} w_{k+l} w_{-l-n}, \qquad k, l \in \mathbb{Z}.$$

Consequently, for any $\lambda \in \mathfrak{U}_n$

$$\|g_+\mathrm{e}_{-n}\|_{\ell^{w,1}} \leqslant \sum_{k,l\neq n} \frac{\langle k-n\rangle}{\langle k+l\rangle|n-k|\langle l+n\rangle|n-l|} w_{k+l}|\varphi_{k+l}^+| w_{-l-n}|f_l^-|,$$

and with Cauchy-Schwarz and Young's inequality for convolutions (B.1),

$$\leqslant \left(\sum_{k,l\neq n}\frac{\langle k-n\rangle^2}{\langle k+l\rangle^2|n-k|^2\langle l+n\rangle^2|n-l|^2}\right)^{1/2}\|\varphi\|_w\|f_-\|_{w;-n}.$$

One further checks that

$$\sup_{l\neq n}\sum_{k\neq n}\frac{\langle k-n\rangle^2}{\langle k+l\rangle^2|n-k|^2} \leqslant 32/5, \qquad \sum_{l\neq n}\frac{1}{\langle l+n\rangle^2|n-l|^2} \leqslant \frac{5/2}{\langle n\rangle^2}.$$

Hence, we obtain for $\|g_+\mathrm{e}_{-n}\|_{\ell^{w,1}}$ and similarly for $\|g_-\mathrm{e}_n\|_{\ell^{w,1}}$,

$$\|g_+\mathrm{e}_{-n}\|_{\ell^{w,1}} \leqslant \frac{4}{\langle n\rangle}\|\varphi\|_w\|f_-\|_{w;-n}, \quad \|g_-\mathrm{e}_n\|_{\ell^{w,1}} \leqslant \frac{4}{\langle n\rangle}\|\varphi\|_w\|f_+\|_{w;n}.$$

Finally, with $\|T_n^2 f\|_{w;n} = \|\Phi g\|_{w;n}$, this gives

$$\|T_n^2 f\|_{w;n}^2 \leqslant \|\varphi\|_w^2\left(\|g_+\mathrm{e}_{-n}\|_{\ell^{w,1}}^2 + \|g_-\mathrm{e}_n\|_{\ell^{w,1}}^2\right) \leqslant \frac{16}{\langle n\rangle^2}\|\varphi\|_w^4\|f\|_{w;n}^2. \quad \blacksquare$$

Consequently, if $w \in \mathcal{M}^1$, then T_n^2 is a $1/2$-contraction for $\langle n\rangle \geqslant 8\|\varphi\|_w^2$. We can now proceed as in Section 6 and find a unique solution

$$\Phi v = \hat{T}_n T_n \Phi u$$

of the Q-equation, which, when inserted into the P-equation, yields

$$A_\lambda u = P_n(I + \hat{T}_n T_n)\Phi u = P_n \hat{T}_n \Phi u.$$

The P-equation is thus equivalent to

$$S_n u = 0, \qquad S_n\colon \mathcal{P}_n \to \mathcal{P}_n, \quad u \mapsto (A_\lambda - P_n \hat{T}_n \Phi)u,$$

where for S_n we have the representation

$$A_\lambda = \begin{pmatrix} \lambda - n\pi & \\ & \lambda - n\pi \end{pmatrix}, \qquad P_n \hat{T}_n \Phi = \begin{pmatrix} a_n & b_n^+ \\ b_n^- & a_n \end{pmatrix}.$$

The the coefficients of the latter matrix are given by

$$\begin{aligned} a_n &= \langle T_n(\mathrm{Id} - T_n^2)^{-1}\Phi e_n^+, e_n^+\rangle = \langle T_n(\mathrm{Id} - T_n^2)^{-1}\Phi e_n^-, e_n^-\rangle, \\ b_n^\pm &= \varphi_{2n}^\pm + \langle T_n^2(\mathrm{Id} - T_n^2)^{-1}\Phi e_n^\mp, e_n^\pm\rangle, \end{aligned} \tag{5.7}$$

depend analytically on λ and φ, and reflect certain symmetries of the Fourier coefficients of φ – see Lemma 6.7 and Lemma 6.8. Making use of the fact that the weight w contains a factor growing linearly in n, we obtain the following improved version of Lemma 6.8

Lemma 25.3 ([59]) *If $\varphi \in H_c^w$ with $w \in \mathcal{M}^1$, then for any $\langle n\rangle \geqslant 8\|\varphi\|_w^2$ the coefficients a_n and $b_n^\pm$ are analytic functions on $\mathfrak{U}_n$ with bounds*

$$|a_n|_{\mathfrak{U}_n} \leqslant \frac{1}{\langle n\rangle}\|\varphi\|_w^2, \qquad w_{2n}|b_n^\pm - \varphi_{2n}^\pm|_{\mathfrak{U}_n} \leqslant \frac{8}{\langle n\rangle}\|\varphi\|_w^2\|\varphi_\pm\|_w. \quad \rtimes$$

Proof. Since $\|T_n^2\|_{w;n} \leqslant 1/2$, the series expansions of a_n and $b_n^\pm$ converge uniformly on $\mathfrak{U}_n$ to analytic functions. Let $u = (I - T_n^2)^{-1}\Phi e_n^+$, then

$$\|u\|_{w;n} \leqslant \|(I - T_n^2)^{-1}\|_{w;n}\|\Phi e_n^+\|_{w;n} \leqslant 2\|\varphi_- \mathrm{e}_n\|_{w;-n} = 2\|\varphi_-\|_w.$$

With the series expansion $u = \sum_{m\in\mathbb{Z}} u_m e_m^-$ we may write

$$a_n = \langle T_n u, e_n^+\rangle = \sum_{m\neq n} \frac{\varphi_{n+m}^-}{\lambda - m\pi} u_m.$$

Since $|n-m| \leqslant |n|$ implies $|n+m| \geqslant 2|n| - |n-m| \geqslant |n|$, this gives

$$\begin{aligned} |a_n|_{\mathfrak{U}_n} &\leqslant \sum_{m\neq n} \frac{1}{\langle n+m\rangle^2 |n-m|} w_{n+m}|\varphi_{n+m}^-| \cdot w_{n+m}|u_m| \\ &\leqslant \frac{1}{\langle n\rangle}\|\varphi_-\|_w\|u\|_{w;n} \leqslant \frac{2}{\langle n\rangle}\|\varphi_-\|_w^2. \end{aligned}$$

Similarly, using the representation $a_n = \langle T_n(I - T_n^2)\Phi e_n^-, e_n^-\rangle$ instead of $a_n = \langle T_n(I - T_n^2)\Phi e_n^+, e_n^+\rangle$ gives

$$|a_n|_{\mathfrak{U}_n} \leqslant \frac{2}{\langle n\rangle}\|\varphi_+\|_w^2.$$

Summing both estimates up gives the first bound. To obtain the second bound, note that one has $b_n^- - \varphi_{2n}^- = \langle T_n^2 u, e_n^-\rangle$ in view of (5.7). Since $\langle f, e_n^-\rangle = \langle f\mathrm{e}_{-n}, e_{2n}^-\rangle$ for any function $f \in L_c^2$, we conclude

$$w_{2n}|\langle T_n^2 u, e_n^-\rangle| \leqslant \|T_n^2 u\|_{w;n} \leqslant \frac{8}{\langle n\rangle}\|\varphi\|_w^2\|\varphi_-\|_w.$$

The proof for b_n^+ is analogous. ∎

In consequence, we obtain the following refinement of Lemma 6.9 describing the roots of $\det S_n$.

Lemma 25.4 ([59]) *Let $\varphi \in H^1_c$, then for any $\langle n\rangle \geqslant 8\|\varphi\|^2_{H^1}$, the determinant of S_n has exactly two complex roots in $\mathfrak{U}_n$ which coincide with the periodic eigenvalues λ_n^-, λ_n^+, are contained in the disc*

$$D_n^\star(\varphi) = \left\{\lambda \in \mathbb{C} : |\lambda - \pi n| \leqslant \frac{\|\varphi\|^2_{H^1}}{\langle n\rangle} + \frac{\sqrt{2}\|\varphi\|_{H^1}}{\langle 2n\rangle}\right\} \subset \left\{\lambda \in \mathbb{C} : |\lambda - n\pi| \leqslant \frac{\pi}{5}\right\},$$

and satisfy

$$|\lambda_n^+ - \lambda_n^-|^2 \leqslant 6|b_n^+ b_n^-|_{\mathfrak{U}_n}. \quad \rtimes$$

Proof. The estimates of the preceding lemma give for $\langle n\rangle \geqslant 8\|\varphi\|^2_{H^1}$,

$$|a_n|_{\mathfrak{U}_n} \leqslant \frac{\|\varphi\|^2_{H^1}}{\langle n\rangle}, \qquad \langle 2n\rangle^2 |b_n^+ b_n^-|_{\mathfrak{U}_n} \leqslant \frac{1}{2}\left(1 + \frac{8}{\langle n\rangle}\|\varphi\|^2_{H^1}\right)^2 \left(\|\varphi_+\|^2_{H^1} + \|\varphi_-\|^2_{H^1}\right),$$

where we used $\langle 2n\rangle |b_n^\pm| \leqslant \|\varphi_\pm\|_{H^1} + \langle 2n\rangle |b_n^\pm - \varphi^\pm_{2n}|$. Therefore,

$$|a_n|_{\mathfrak{U}_n} + |b_n^+ b_n^-|^{1/2}_{\mathfrak{U}_n} \leqslant \inf_{\lambda \in \mathfrak{U}_n \setminus D_n} |\lambda - n\pi| = \frac{\|\varphi\|^2_{H^1}}{\langle n\rangle} + \sqrt{2}\frac{\|\varphi\|_{H^1}}{\langle 2n\rangle} \leqslant \pi/5.$$

It follows from Rouche's Theorem that the function $h = \lambda - n\pi - a_n$ has a single root in $D_n^\star(\varphi)$, just as $\lambda - n\pi$. Furthermore, h^2 and $\det S_n$ have the same number of roots in $D_n^\star(\varphi)$, namely two when counted with multiplicity, while $\det S_n$ clearly has no root in $\mathfrak{U}_n \setminus D_n^\star(\varphi)$. As observed in Lemma 6.6, a complex number $\lambda \in \mathfrak{U}_n$ is a periodic eigenvalue of φ if and only if it is a root of $\det S_n$. Moreover, the strips $\mathfrak{U}_n$ cover the complex plane and $\lambda_n^\pm = n\pi + \ell_n^2$, hence it follows from a simple counting argument that the roots of $\det S_n$ are precisely the eigenvalues $\lambda_n^\pm$. The estimate of $|\lambda_n^+ - \lambda_n^-|$ follows in the same way as in Lemma 6.9. ∎

Proof of Theorem 25.1. For each $\langle n\rangle \geqslant 8\|\varphi\|^2_{H^1}$, Lemma 6.9 applies giving $\lambda_n^\pm \in D_n^\star(\varphi)$. In turn, the remaining eigenvalues have to be contained in the strip

$$\mathbb{C} \setminus \left(\bigcup_{\langle n\rangle \geqslant 8\|\varphi\|^2_{H^1}} \mathfrak{U}_n\right) \subset \left\{\lambda \in \mathbb{C} : |\Re\lambda| \leqslant (8\|\varphi\|^2_{H^1} - 1/2)\pi\right\}.$$

To obtain the estimate for the imaginary part, suppose f is a L^2_c normalized eigenfunction for λ, then

$$2\Im\lambda = \lambda - \bar{\lambda} = \langle Lf, f\rangle - \langle f, Lf\rangle = \langle (L - L^*)f, f\rangle.$$

Further, using the L^∞-estimate $\|g\|_{L^\infty} \leqslant \sqrt{2}\|g\|_{H^1}$ for $g \in H^1$, we find

$$\|(L - L^*)f\|_{L^2} \leqslant \sqrt{2}\|\varphi_+ - \overline{\varphi_-}\|_{H^1}\|f\|_{L^2} \leqslant 2\|\varphi\|_{H^1}.$$

This completes the proof of the theorem. ∎

Incidentally, we obtain the following estimate for the gap lengths, which will be used in Section 28.

Proposition 25.5 ([59]) *Suppose $\varphi \in H^w_c$ with $w \in \mathcal{M}^1$, then for any $\langle N\rangle \geqslant 8\|\varphi\|^2_w$,*

$$\sum_{|n|\geqslant N} w_{2n}^2 |\gamma_n(\varphi)|^2 \leqslant 6\|R_N\varphi\|^2_w + \frac{1152}{\langle N\rangle}\|\varphi\|^6_w,$$

where $R_N\varphi = \sum_{|n|\geqslant N}(\varphi^-_{2n}\mathrm{e}_{-2n}, \varphi^+_{2n}\mathrm{e}_{2n})$. If, in addition, φ is in the complex neighborhood $\mathcal{W}$ of L^2_r, then

$$\sum_{n\in\mathbb{Z}} w_{2n}^2 |\gamma_n(\varphi)|^2 \leqslant 265\pi^2 w^2[16\|\varphi\|^2_w](1 + \|\varphi\|^2_w)\|\varphi\|^2_w. \quad \rtimes$$

Proof. By Lemma 6.9 we have for any $\langle n\rangle \geqslant 8\|\varphi\|_w^2$ the estimate

$$|\gamma_n(\varphi)|^2 = |\lambda_n^+ - \lambda_n^-|^2 \leqslant 6|b_n^+ b_n^-|_{\mathfrak{U}_n} \leqslant 3(|b_n^+|_{\mathfrak{U}_n}^2 + |b_n^-|_{\mathfrak{U}_n}^2).$$

Using $|b_n^\pm|_{\mathfrak{U}_n} \leqslant |\varphi_{2n}^\pm| + |b_n^\pm - \varphi_{2n}^\pm|_{\mathfrak{U}_n}$ we thus find for any $\langle N\rangle \geqslant 8\|\varphi\|_w^2$,

$$\frac{1}{6}\sum_{|n|\geqslant N} w_{2n}^2|\gamma_n(\varphi)|^2 \leqslant \sum_{|n|\geqslant N} w_{2n}^2(|\varphi_{2n}^+|^2 + |\varphi_{2n}^-|^2 + |b_n^+ - \varphi_{2n}^+|_{\mathfrak{U}_n}^2 + |b_n^- - \varphi_{2n}^-|_{\mathfrak{U}_n}^2).$$

Further by Lemma 6.8, $w_{2n}|\varphi_{2n}^\pm - b_n^\pm|_{\mathfrak{U}_n} \leqslant 8\langle n\rangle^{-1}\|\varphi\|_w^2\|\varphi_\pm\|_w$, hence

$$\frac{1}{6}\sum_{|n|\geqslant N} w_{2n}^2|\gamma_n(\varphi)|^2 \leqslant \|R_{2N}\varphi\|_w^2 + 64\|\varphi\|_w^6 \sum_{|n|\geqslant N}\frac{1}{\langle n\rangle^2},$$

and the first claim follows with $\sum_{|n|\geqslant N} 1/\langle n\rangle^2 \leqslant 3/\langle N\rangle$.

If additionally $\varphi \in \mathcal{W}$, then each gap is contained in its isolating neighborhood U_n. Those are disjoint complex discs centered on the real line, whose diameters for $|n| < N$ sum up to at most $(2N-1)\pi$ by Theorem 25.1. Therefore,

$$\sum_{|n|<N} w_{2n}^2|\gamma_n(\varphi)|^2 \leqslant w_{2N-2}^2\Bigl(\sum_{|n|<N}|\gamma_n(\varphi)|\Bigr)^2 \leqslant w_{2N-2}^2(2N-1)^2\pi^2,$$

and choosing $N+1 \geqslant 8\|\varphi\|_w^2 > N$ gives the second claim. ∎

If, for $\langle n\rangle \geqslant 8\|\varphi\|_w^2$, we use

$$w_{2n}|b_n^\pm| \leqslant \|\varphi_\pm\|_w + \frac{8}{\langle n\rangle}\|\varphi\|_w^2\|\varphi_\pm\| \leqslant 2\|\varphi_\pm\|_w,$$

then we obtain the *individual gap estimate*

$$w_{2n}|\gamma_n(\varphi)| \leqslant \sqrt{12}\|\varphi\|_w. \tag{5.8}$$

26. Estimating the Actions

As an immediate corollary to the localization obtained in the previous section, we obtain the following quantitative estimate of the high-level actions.

Proposition 26.1 ([59]) *If $\varphi \in H_r^1$, then for $\langle n\rangle \geqslant 8\|\varphi\|_{H^1}^2$ and $m \geqslant 0$,*

$$J_{n,2m+1} = \zeta_{n,m}^{2m} I_n, \qquad |\zeta_{n,m} - n\pi| \leqslant \frac{\|\varphi\|_{H^1}^2}{\langle n\rangle} + \frac{\sqrt{2}\|\varphi\|_{H^1}}{\langle 2n\rangle}.$$

In particular, if $n \neq 0$ and $\langle n\rangle \geqslant 8\|\varphi\|_{H^1}^2$, then

$$2^{-m}\langle 2n\pi\rangle^{2m} I_n \leqslant 4^m J_{n,2m+1} \leqslant \langle 2n\pi\rangle^{2m} I_n,$$

while the remaining actions for all $\langle n\rangle < 8\|\varphi\|_{H^1}^2$ satisfy

$$4^m|J_{n,2m+1}| \leqslant (16\pi)^{2m}\|\varphi\|_{H^1}^{4m} I_n. \quad \rtimes$$

Proof. Recall from (5.6) that $J_{n,2m+1} = \zeta_{n,m}^{2m} I_n$ with $\zeta_{n,m} \in G_n$. Provided $\langle n\rangle \geqslant 8\|\varphi\|_{H^1}^2$, the estimate of $|\zeta_{n,m} - n\pi|$ follows from Theorem 25.1. If additionally $n \neq 0$, then $\langle 2n\rangle \geqslant 3\langle n\rangle/2$ and hence $|\zeta_{n,m} - n\pi| \leqslant 1/2$. In consequence,

$$\frac{1}{\sqrt{2}}\langle 2n\pi\rangle \leqslant 2|\zeta_{n,m}| \leqslant \langle 2n\pi\rangle, \qquad n \neq 0.$$

Conversely, if $\langle n\rangle < 8\|\varphi\|_{H^1}^2$, then $|\zeta_{n,m}| \leqslant 8\pi\|\varphi\|_{H^1}^2$. ∎

In the sequel we use Proposition 26.1 to obtain an estimate of

$$\|I(\varphi)\|_{\ell^{2m,1}} = \sum_{n\in\mathbb{Z}} \langle 2n\pi\rangle^{2m} |I_n|$$

in terms of $\sum_{n\in\mathbb{Z}} J_{n,2m+1}$ and a remainder depending solely on $\|\varphi\|_{H^1}$. The trace formula and the polynomial structure of the Hamiltonians then allows us to obtain the first part of Theorem 23.2.

Lemma 26.2 ([59]) *For every $m \geqslant 1$,*

$$\|I(\varphi)\|_{\ell^{2m,1}} \leqslant \langle 16\pi\rangle^{2m}(1+\|\varphi\|_{H^1})^{4m}\|\varphi\|_{L^2}^2 + (-1)^{m+1}2^m\mathcal{H}_{2m+1}(\varphi),$$

uniformly for all $\varphi \in H_r^m$. ⋊

Proof. Choose $N+1 \geqslant 8\|\varphi\|_{H^1}^2 > N$, then by Proposition 26.1, the trace formula (5.4), and the positivity of the actions

$$\sum_{|n|>N} \langle 2n\pi\rangle^{2m} I_n \leqslant 8^m \sum_{n\in\mathbb{Z}} J_{n,2m+1} = (-1)^{m+1}2^m\mathcal{H}_{2m+1}.$$

On the other hand, by our choice of N and the trace formula (5.3)

$$\sum_{|n|\leqslant N} \langle 2n\pi\rangle^{2m} I_n \leqslant \langle 2N\pi\rangle^{2m} \sum_{n\in\mathbb{Z}} I_n \leqslant \langle 16\pi\rangle^{2m}(1+\|\varphi\|_{H^1})^{4m}\|\varphi\|_{L^2}^2. \qquad \blacksquare$$

We denote the two components of $\varphi \in L_r^2$ by $\varphi = (\psi, \overline{\psi})$, and write $\psi_{(m)} = \partial_x^m \psi$ to simplify notation such that

$$\int_{\mathbb{T}} |\psi_{(m)}|^2 \,\mathrm{d}x = \frac{1}{2}\|\varphi_{(m)}\|_{L^2}^2.$$

We further note that on H_r^m,

$$\mathcal{H}_{2m+1}(\varphi) = \int_{\mathbb{T}} \Big(|\psi_{(m)}|^2 + p_{2m}(\psi, \overline{\psi}, \dots, \psi_{(m-1)}, \overline{\psi}_{(m-1)}) \Big) \,\mathrm{d}x,$$

with p_{2m} being a homogenous polynomial of degree $2m+2$ when ψ, $\overline{\psi}$, and ∂_x each count as one degree. Further, the degree of each monomial of p_{2m} with respect to ψ equals the degree with respect to $\overline{\psi}$ – see Corollary H.2 from the appendix. Consequently, each monomial $\mathfrak{q}$ of p_{2m} may be estimated by

$$|\mathfrak{q}| \leqslant c_{\mathfrak{q}} |\psi|^{\mu_0} |\psi_x|^{\mu_1} \cdots |\psi_{(m-1)}|^{\mu_{m-1}},$$

with some positive constant $c_{\mathfrak{q}}$ and integers $\mu_0, \dots, \mu_{m-1}$. Since $\mathfrak{q}$ has degree $2m+2$, we have

$$\sum_{0\leqslant i\leqslant m-1} (1+i)\mu_i = 2m+2,$$

and as the degree with respect to ψ and $\overline{\psi}$ is the same, $\sum_{0\leqslant i\leqslant m-1} \mu_i$ is an even integer. Denote by $\mathcal{I}_{2m+2} \subset (\mathbb{Z}_{\geqslant 0}/2)^m$ the set of all multi-indices $\mu = (\mu_i)_{0\leqslant i\leqslant m-1}$ satisfying the constraints

$$\sum_{0\leqslant i\leqslant m-1} (1+i)\mu_i = 2m+2, \qquad |\mu| := \sum_{0\leqslant i\leqslant m-1} \mu_i \in 2\mathbb{Z}. \tag{5.9}$$

Then, we obtain the estimate

$$|p_{2m}| \leqslant \sum_{\mu\in\mathcal{I}_{2m+2}} c_\mu |\mathfrak{q}_\mu|, \qquad |\mathfrak{q}_\mu| = |\psi|^{\mu_0} |\psi_x|^{\mu_1} \cdots |\psi_{(m-1)}|^{\mu_{m-1}}, \tag{5.10}$$

with positive constants c_μ. This representation of p_{2m} allows us to obtain detailed estimates of the Hamiltonians $\mathcal{H}_{2m+1}$, which improves on the ones obtained in [59] in the sense that the remainders involve only L^2-norms instead of H^{m-1}-norms.

Lemma 26.3 *For any $m \geqslant 1$ and any $\varepsilon > 0$ there exists $C_{\varepsilon,m}$ so that*

$$\int_{\mathbb{T}} |\mathfrak{p}_{2m}| \, \mathrm{d}x \leqslant \varepsilon \|\partial_x^m \psi\|_{L^2}^2 + C_{\varepsilon,m}(1 + \|\psi\|_{L^2}^{4m})\|\psi\|_{L^2}^2.$$

In particular,

$$|\mathcal{H}_{2m+1}| \leqslant (1+\varepsilon)\|\partial_x^m \psi\|_{L^2}^2 + C_{\varepsilon,m}(1 + \|\psi\|_{L^2}^{4m})\|\psi\|_{L^2}^2. \quad \rtimes$$

Proof. First note that $|\mu| \geqslant 4$ for any $\mu \in \mathcal{I}_{2m+2}$. Indeed, under the constraint $|\mu| = k$ for some fixed $k \geqslant 0$, the expression $\sum_{0 \leqslant i \leqslant m-1}(1+i)\mu_i$ attains its maximum when $\mu_{m-1} = |\mu|$ while all other coefficients vanish. In this case,

$$\sum_{0 \leqslant i \leqslant m-1} (1+i)\mu_i = m|\mu|,$$

and the right hand side is strictly less than $2m+2$ if $|\mu| \leqslant 2$. Therefore, $|\mu|$ is an even integer strictly larger than 2.

As a consequence, for any $\mu \in \mathcal{I}_{2m+2}$, there either exist two distinct nonzero coefficients μ_k, μ_l with $0 \leqslant k, l \leqslant m-1$ or $\mu_k = |\mu| \geqslant 4$ for some $0 \leqslant k \leqslant m-1$ while all other coefficients vanish. In the first case, using Cauchy-Schwarz and the L^∞-estimate, we obtain

$$\begin{aligned}
\int_{\mathbb{T}} |\mathfrak{q}_\mu| \, \mathrm{d}x &\leqslant \int \left(\prod_{0 \leqslant i \neq k,l}^{m-1} |\psi_{(i)}|^{\mu_i} \right) |\psi_{(k)}|^{\mu_k - 1} |\psi_{(l)}|^{\mu_l - 1} |\psi_{(k)}| |\psi_{(l)}| \, \mathrm{d}x \\
&\leqslant \left(\prod_{0 \leqslant i \neq k,l}^{m-1} \|\psi_{(i)}\|_{L^\infty}^{\mu_i} \right) \|\psi_{(k)}\|_{L^\infty}^{\mu_k - 1} \|\psi_{(l)}\|_{L^\infty}^{\mu_l - 1} \|\psi_{(k)}\|_{L^2} \|\psi_{(l)}\|_{L^2}.
\end{aligned}$$

It follows from the generalized Gagliardo-Nierenberg inequality that for any integers $0 \leqslant i \leqslant m$

$$\|\psi_{(i)}\|_{L^\infty} \lesssim \|\psi\|_{H^m}^{\frac{i+1/2}{m}} \|\psi\|_{L^2}^{1 - \frac{i+1/2}{m}}, \qquad \|\psi_{(i)}\|_{L^2} \lesssim \|\psi\|_{H^m}^{\frac{i}{m}} \|\psi\|_{L^2}^{1-\frac{i}{m}}, \tag{5.11}$$

where $a \lesssim b$ means $a \leqslant c \cdot b$ with c being a multiplicative constant which is independent of ψ and only depends on the parameters i and m. We thus obtain

$$\int_{\mathbb{T}} |\mathfrak{q}_\mu| \, \mathrm{d}x \lesssim \|\psi\|_{H^m}^{\left(\sum_{i=0}^{m-1} \frac{i+1/2}{m}\mu_i\right) - \frac{1}{m}} \|\psi\|_{L^2}^{\left(\sum_{i=0}^{m-1}\left(1 - \frac{i+1/2}{m}\right)\mu_i\right) + \frac{1}{m}}. \tag{5.12}$$

In the second case where $\mu_k = |\mu| \geqslant 4$ while all other coefficients of μ vanish, we get

$$\begin{aligned}
\int_{\mathbb{T}} |\mathfrak{q}_\mu| \, \mathrm{d}x &\leqslant \int \left(\prod_{0 \leqslant i \neq k,l}^{m-1} |\psi_{(i)}|^{\mu_i} \right) |\psi_{(k)}|^{\mu_k - 2} |\psi_{(k)}|^2 \, \mathrm{d}x \\
&\leqslant \left(\prod_{0 \leqslant i \neq k,l}^{m-1} \|\psi_{(i)}\|_{L^\infty}^{\mu_i} \right) \|\psi_{(k)}\|_{L^\infty}^{\mu_k - 2} \|\psi_{(k)}\|_{L^2}^2.
\end{aligned}$$

The interpolation inequality (5.11) then yields estimate (5.12) also in this case.

Recall from (5.9) that $\sum_{i=0}^{m-1}(i+1/2)\mu_i = 2m+2-|\mu|/2$, hence

$$\sum_{i=0}^{m-1} \frac{i+1/2}{m}\mu_i - \frac{1}{m} = \frac{2m+2-|\mu|/2}{m} - \frac{1}{m} = 2 - \frac{|\mu|-2}{2m} < 2,$$

where in the last line we used that $4 \leqslant |\mu| \leqslant 2m+2$. Similarly,

$$\sum_{i=0}^{m-1} \left(1 - \frac{i+1/2}{m}\right)\mu_i + \frac{1}{m} = |\mu| - 2 + \frac{|\mu|-4}{2m} = \left(1 + \frac{1}{2m}\right)(|\mu|-2).$$

Both identities together yield in view of (5.12)

$$\int |\mathfrak{q}_\mu| \, \mathrm{d}x \lesssim \|\psi\|_{H^m}^{2-\frac{|\mu|-2}{2m}} \|\psi\|_{L^2}^{\left(1+\frac{1}{2m}\right)(|\mu|-2)}.$$

Applying Young's inequality to the latter with

$$p = \frac{2}{2-\frac{|\mu|-2}{2m}}, \qquad p' = \frac{4m}{|\mu|-2},$$

then finally gives for any $\varepsilon > 0$

$$\begin{aligned}\int_{\mathbb{T}} |\mathfrak{q}_\mu| \, \mathrm{d}x &\leqslant \varepsilon \|\psi\|_{H^m}^{\left(2-\frac{|\mu|-2}{2m}\right)p} + C_{\varepsilon,m} \|\psi\|_{L^2}^{\left(1+\frac{1}{2m}\right)(|\mu|-2)p'} \\ &= \varepsilon \|\psi\|_{H^m}^2 + C_\varepsilon \|\psi\|_{L^2}^{4m+2},\end{aligned}$$

where $C_{\varepsilon,m}$ is an absolute constant independent of ψ. Since this estimate holds for any monomial of p_{2m}, the final claim follows with the fact that

$$\|\psi\|_{H^m}^2 \leqslant 2^{m-1} \|\partial_x^m \psi\|_{L^2}^2 + 2^{m-1} \|\psi\|_{L^2}^2. \quad \blacksquare$$

Proof of Theorem 23.2 (i) Recall from Lemma 26.2, that

$$\|I(\varphi)\|_{\ell^{2m,1}} \leqslant \langle 16\pi \rangle^{2m} (1 + \|\varphi\|_{H^1})^{4m} \|\varphi\|_{L^2}^2 + (-1)^{m+1} 2^m \mathcal{H}_{2m+1}(\varphi),$$

which together with the estimate of the Hamiltonian from Lemma 26.3 with $\varepsilon = 1$ yields

$$\|I(\varphi)\|_{\ell^1_{2m}} \leqslant 2^m \|\varphi\|_{H^m}^2 + c_m^2 (1 + \|\varphi\|_{H^1})^{4m} \|\varphi\|_{L^2}^2, \qquad m \geqslant 1,$$

where c_m is an absolute constant depending only on m. $\blacksquare$

27. Estimating the Sobolev Norms

We now turn to the converse problem of estimating the Sobolev norms of the potential in terms of weighted norms of its actions on level one. Our starting point is the identity

$$\frac{1}{2} \|\varphi_{(m)}\|_0^2 = 4^m \sum_{n \in \mathbb{Z}} J_{n,2m+1} - \int_{\mathbb{T}} p_{2m} \, \mathrm{d}x, \qquad m \geqslant 1, \tag{5.13}$$

which is obtained by combining Corollary H.2 and the trace formula (5.4). The key step is estimating the actions $J_{n,2m+1}$ in terms of I_n. Subsequently, p_{2m} is estimated by Lemma 26.3. The main difficulty is to estimate the actions $J_{n,2m+1}$ below the threshold of $\langle n \rangle$ provided by Proposition 26.1. Even though there are only finitely many of them, they cannot be controlled by the L^2-norm $\|\varphi\|_{L^2}$ as one can translate the spectrum of φ without changing $\|\varphi\|_{L^2}$. Instead, we use the H^1-norm $\|\varphi\|_{H^1}$, and provide estimates of $\|\varphi\|_{H^1}$ in terms of I_n by separate arguments – see also Korotyaev [45, 48].

Lemma 27.1 ([59]) *Uniformly for all $\varphi \in H_r^1$,*

$$\mathcal{H}_3(\varphi) - 2\mathcal{H}_1^2(\varphi) \leqslant \sum_{n \in \mathbb{Z}} (2n\pi)^2 I_n(\varphi).$$

In particular, $\mathcal{H}_3(\varphi) \leqslant \|I(\varphi)\|_{\ell^{2,1}} + 2\|I(\varphi)\|_{\ell^1}^2$ and

$$\frac{1}{3} \|\varphi\|_{H^1}^2 \leqslant \|I(\varphi)\|_{\ell^{2,1}} + \|I(\varphi)\|_{\ell^1}^2. \quad \rtimes$$

Proof. By Proposition 21.3 and by Lemma 21.1

$$\mathcal{H}_3 - 2\mathcal{H}_1^2 - \sum_{n\in\mathbb{Z}} (2n\pi)^2 I_n = -\frac{4}{3}\sum_{n\in\mathbb{Z}} \mathcal{R}_n^{(3)} \leqslant 0,$$

since all moments $\mathcal{R}_n^{(3)}$, $n \in \mathbb{Z}$, are nonnegative. This proves the first claim. Note that $\langle 2n\pi\rangle^2 \leqslant \frac{3}{2}(1 + (2n\pi)^2)$ for $n \in \mathbb{Z}$, so

$$\frac{1}{3}\|\varphi\|_{H^1}^2 \leqslant \frac{1}{2}(\|\varphi_x\|_{L^2}^2 + \|\varphi\|_{L^2}^2) = \mathcal{H}_3 - \int_{\mathbb{T}} |\psi|^4 \,\mathrm{d}x + \mathcal{H}_1.$$

Since $\int_{\mathbb{T}}|\psi|^4 \,\mathrm{d}x \geqslant \mathcal{H}_1^2$, the second claim follows with

$$\frac{1}{3}\|\varphi\|_{H^1}^2 \leqslant \mathcal{H}_3 - 2\mathcal{H}_1^2 + \mathcal{H}_1^2 + \mathcal{H}_1 \leqslant \|I(\varphi)\|_{\ell^{2,1}} + \|I(\varphi)\|_{\ell^1}^2. \qquad ∎$$

Proof of Theorem 23.2 (ii). The case $m = 1$ is an immediate corollary of the lemma above. For the case $m \geqslant 2$ we find with (5.13)

$$\frac{1}{2}\|\varphi_{(m)}\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} 4^m J_{n,2m+1} - \int_{\mathbb{T}} p_{2m} \,\mathrm{d}x.$$

Choosing $\varepsilon = 1/4$ in Lemma 26.3 then gives

$$\frac{1}{4}\|\varphi_{(m)}\|_{L^2}^2 \lesssim \sum_{n\in\mathbb{Z}} 4^m J_{n,2m+1} + \|\varphi\|_{L^2}^{4m+2} \lesssim \sum_{n\in\mathbb{Z}} 4^m J_{n,2m+1} + \|I(\varphi)\|_{\ell^1}^{2m+1},$$

where we applied the trace formula (5.3) in the last step. Finally, in view of Lemma 27.2 below

$$\frac{1}{4}\|\varphi_{(m)}\|_{L^2}^2 \leqslant \|I\|_{\ell^{2m,1}} + c_m(1 + \|I(\varphi)\|_{\ell^{2,1}})^{4m-3}\|I(\varphi)\|_{\ell^{1(\varphi)}}. \qquad ∎$$

It remains to prove the following lemma used in the proof of Theorem 23.2 (ii).

Lemma 27.2 *For each $m \geqslant 1$,*

$$\mathcal{H}_{2m+1} = \sum_{n\in\mathbb{Z}} 4^m J_{n,2m+1} \leqslant \|I(\varphi)\|_{\ell^{2m,1}} + (64\pi)^{2m-2}(1 + \|I(\varphi)\|_{\ell^{2,1}})^{4m-3}\|I(\varphi)\|_{\ell^{2,1}},$$

uniformly for all $\varphi \in H_r^m$. ⋊

Proof. Let $N + 1 \geqslant 8\|\varphi\|_1^2 > N$, then by Proposition 26.1

$$\sum_{|n|>N} 4^m J_{n,2m+1} \leqslant \sum_{|n|>N} \langle 2n\pi\rangle^{2m} I_n,$$

while for the remaining actions $J_{n,2m+1} = \zeta_{n,m}^{2m-2} J_{n,3}$ and hence

$$\begin{aligned}\sum_{|n|\leqslant N} 4^m J_{n,2m+1} &\leqslant (16\pi)^{2m-2}(1 + \|\varphi\|_{H^1}^2)^{2m-2} \sum_{|n|\leqslant N} 4J_{n,3}\\ &\leqslant (64\pi)^{2m-2}(1 + \|I(\varphi)\|_{\ell^{2,1}})^{4m-4} \sum_{|n|\leqslant N} 4J_{n,3}.\end{aligned}$$

By the trace formula (5.4) and Lemma 27.1 we finally obtain

$$\sum_{n\in\mathbb{Z}} 4J_{n,3} = \mathcal{H}_3 \leqslant \|I(\varphi)\|_{\ell^{2,1}} + 2\|I(\varphi)\|_{\ell^1}^2. \qquad ∎$$

28. Actions, Weighted Sobolev Spaces, and Gap Lengths

The case of estimating the action variables of φ in standard Sobolev spaces H_r^m with integers $m \geqslant 1$ is somewhat special due to the presence of the trace formula (5.4). When arbitrary weighted Sobolev spaces H_r^w are considered, there is no identity known to exist relating $\|\varphi\|_w$ to Hamiltonians of the NLS-hierarchy. Albeit, even in the case of weighted Sobolev spaces, the regularity properties of φ are well known to be closely related to the decay properties of the gap lengths $\gamma_n(\varphi)$ – see e.g. [18, 37] and section 25. Moreover,

$$\frac{4I_n}{\gamma_n^2} = 1 + \ell_n^2 \tag{5.14}$$

holds locally uniformly on L_r^2 and hence uniformly on bounded subsets of H_r^1 – see Theorem 11.2. In this section we prove a quantitative version of (5.14) which is quadratic in $\|\varphi\|_{H^1}$ on all of H_r^1. From this and the estimates of the gap lengths given in Section 25 we then obtain Theorem 23.4.

To set the stage, let $\varphi \in \mathcal{W}$ and recall from the definition of the actions (2.16) and the product representation of the action integrand (2.15) – see also (2.18)

$$I_n = -\frac{1}{\pi}\int_{\Gamma_n} \frac{(\lambda_n^\bullet - \lambda)\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}}\,\mathrm{d}\lambda = \frac{\mathrm{i}}{\pi}\int_{\Gamma_n} \frac{(\lambda_n^\bullet - \lambda)^2}{w_n(\lambda)}\chi_n(\lambda)\,\mathrm{d}\lambda, \qquad \chi_n(\lambda) = \prod_{n\neq m} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}.$$

In the case $I_n \neq 0$, or equivalently $\gamma_n \neq 0$, we shrink the contour Γ_n to the straight line $G_n = [\lambda_n^-, \lambda_n^+]$ and use the parametrization $\lambda_t = \tau_n + t\gamma_n/2$ of G_n together with $w_n|_{G_n^\pm}(\lambda_t) = \mp\mathrm{i}\frac{\gamma_n}{2}\sqrt[+]{1-t^2}$ from (2.12) to obtain

$$\frac{4I_n}{\gamma_n^2} = \frac{2}{\pi}\int_{-1}^{1} \frac{(t-t_n)^2}{\sqrt[+]{1-t^2}}\chi_n(\tau_n + t\gamma_n/2)\,\mathrm{d}t, \qquad t_n = 2(\lambda_n^\bullet - \tau_n)/\gamma_n.$$

By Lemma F.1 there exists an open connected neighborhood $\tilde{\mathcal{W}} \subset \mathcal{W}$ of L_r^2 so that $|\lambda_m^\bullet - \tau_m| \leqslant |\gamma_m|$ for all $m \in \mathbb{Z}$, hence $|t_n| \leqslant 2$, and thus

$$\left|\frac{4I_n}{\gamma_n^2}\right| \leqslant 9 \sup_{\lambda\in G_n} |\chi_n(\lambda)|. \tag{5.15}$$

To get a quantitative version of (5.14) we thus need a uniform estimate of χ_n .

Lemma 28.1 ([59]) *On $H_c^1 \cap \tilde{\mathcal{W}}$ for any $\langle n\rangle \geqslant \langle N\rangle \geqslant 8\|\varphi\|_{H^1}^2$,*

$$|\chi_n|_{[\lambda_n^-,\lambda_n^+]} \leqslant 128\frac{n+N+2/5}{n-N+3/10} \leqslant 2048(1+\|\varphi\|_{H^1}^2). \quad \rtimes$$

Proof. Suppose $\varphi \in H_c^1 \cap \tilde{\mathcal{W}}$ and choose $\langle N\rangle \geqslant 8\|\varphi\|_{H^1}^2 > N$ and $|n| \geqslant N$. We split the product χ_n into two parts,

$$\chi_n(\lambda) = \left(\prod_{|m|<N} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}\right)\left(\prod_{\substack{n\neq m\\ |m|\geqslant N}} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}\right),$$

and first consider the latter factor. Therefore, suppose $|m| \geqslant N$, then $|\lambda_m^\pm - m\pi| \leqslant \pi/5$ by Theorem 25.1, and for any $\lambda \in G_n$,

$$|\tau_m - \lambda| \geqslant |m\pi - n\pi| - |\tau_m - m\pi| - |\lambda - n\pi| \geqslant \frac{3}{5}\pi|n-m|.$$

Further, $|\gamma_m| \leqslant \sqrt{12}\|\varphi\|_{H^1}/\langle 2m\rangle$ by the individual gap estimate (5.8), so that

$$\left|\frac{\gamma_m/2}{\tau_m - \lambda}\right| \leqslant \frac{\|\varphi\|_{H^1}}{\langle 2m\rangle|n-m|} \leqslant \frac{1}{\sqrt{8}}.$$

In particular, $|w_m(\lambda)| \geqslant |\tau_m - \lambda| - |\gamma_m/2|$. Moreover, $|\lambda^\bullet_m - \tau_m| \leqslant |\gamma_m|$ by Lemma F.1. Thus, with $(1+2r)/(1-r) \leqslant 1+5r$ for $0 \leqslant r \leqslant 1/\sqrt{8}$, we conclude

$$\left|\frac{\lambda^\bullet_m - \lambda}{w_m(\lambda)}\right| \leqslant \frac{|\tau_m - \lambda| + |\gamma_m|}{|\tau_m - \lambda| - |\gamma_m|/2} \leqslant 1 + \frac{5}{2}\frac{|\gamma_m|}{|\tau_m - \lambda|} \leqslant 1 + \frac{4}{3}\frac{|\gamma_m|}{|m-n|}.$$

It follows with Cauchy-Schwarz that

$$\begin{aligned}\sum_{\substack{m\neq n\\|m|\geqslant N}} \frac{|\gamma_m|}{|n-m|} &\leqslant \left(\sum_{\substack{m\neq n\\|m|\geqslant N}} \frac{1}{\langle 2m\rangle^2|n-m|^2}\right)^{1/2}\left(\sum_{|m|\geqslant N}\langle 2m\rangle^2|\gamma_m|^2\right)^{1/2}\\ &\leqslant \frac{2}{\langle 2N\rangle}\left(\sum_{|m|\geqslant N}\langle 2m\rangle^2|\gamma_m|^2\right)^{1/2},\end{aligned}$$

and by Proposition 25.5

$$\sum_{|m|\geqslant N}\langle 2m\rangle^2|\gamma_m|^2 \leqslant 6\|R_N\varphi\|_{H^1}^2 + 144\|\varphi\|_{H^1}^4 \leqslant 3\langle N\rangle^2.$$

Therefore, by the infinite product estimate provided in Lemma D.1,

$$\prod_{\substack{|m|\geqslant N\\m\neq n}}\left|\frac{\lambda^\bullet_m - \lambda}{w_m(\lambda)}\right| \leqslant \exp\left(\frac{4}{3}\sum_{\substack{m\neq n\\|m|\geqslant N}}\frac{|\gamma_m|}{|n-m|}\right) \leqslant \exp\left(\frac{8}{\sqrt{3}}\frac{\langle N\rangle}{\langle 2N\rangle}\right) \leqslant 128.$$

It remains to estimate the part of the product $\prod_{\substack{|m|<N\\m\neq n}}\left|\frac{\lambda^\bullet_m-\lambda}{w_m(\lambda)}\right|$. By assumption $|n| \geqslant N$ and the cases $n \geqslant N$ and $n \leqslant -N$ are treated similarly, thus we concentrate on $n \geqslant N$. We note that $\lambda^\bullet_m$ and $\lambda^\pm_m$ are contained in the isolating neighborhood U_m, which is a complex disc centered on the real line. Therefore, for any $\lambda \in G_n$,

$$|\lambda^\pm_{m-1} - \lambda| > |\lambda^\bullet_m - \lambda|, \qquad m < N. \tag{5.16}$$

Consequently,

$$\prod_{|m|<N}\left|\frac{\lambda^\bullet_m - \lambda}{w_m(\lambda)}\right| = \prod_{|m|<N}\left|\frac{\lambda^\bullet_m - \lambda}{w_{m-1}(\lambda)}\right|\left|\frac{w_{-N}(\lambda)}{w_{N-1}(\lambda)}\right| \leqslant \left|\frac{w_{-N}(\lambda)}{w_{N-1}(\lambda)}\right|.$$

Since $|\lambda - n\pi| \leqslant \pi/5$, $|\lambda^\pm_N - N\pi| \leqslant \pi/$, and $|\Re\lambda_{N-1}| \leqslant (N-1/2)\pi$, we conclude

$$\left|\frac{w_{-N}(\lambda)}{w_{N-1}(\lambda)}\right| \leqslant \frac{(N+1/5)\pi + (n+1/5)\pi}{(n-1/5)\pi - (N-1/2)\pi} = \frac{n+N+2/5}{n-N+3/10},$$

and one checks that

$$\frac{n+N+2/5}{n-N+3/10} = 1 + \frac{2N+2/5-3/10}{n-N+3/10} \leqslant \frac{4}{3} + 2N \leqslant 2\langle N\rangle.$$

Altogether we thus have for any $n \geqslant N$,

$$\sup_{\lambda\in G_n}|\chi_n(\lambda)| \leqslant 128\frac{n+N+2/5}{n-N+3/10} \leqslant 256\langle N\rangle \leqslant 2048(1+\|\varphi\|_{H^1}^2). \quad \blacksquare$$

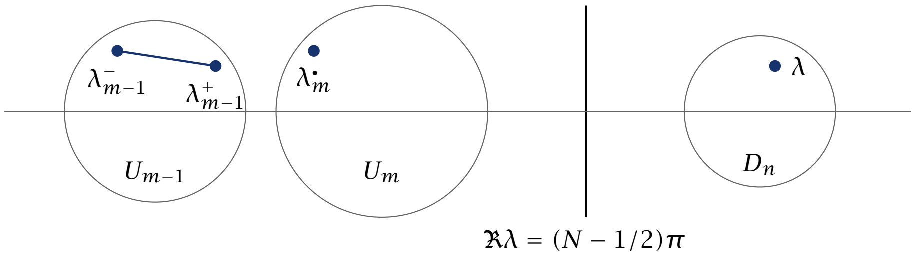

Figure 5.1.: Isolating neighborhoods with $m < N$ and $n \geqslant N$ as in (5.16).

Proposition 28.2 ([59]) *Suppose $\varphi \in H_c^1 \cap \tilde{\mathcal{W}}$, then for any $|n| \geqslant 8\|\varphi\|_{H^1}^2$,*

$$|I_n| \leqslant 4608(1 + \|\varphi\|_{H^1}^2)|\gamma_n|^2. \quad \rtimes$$

Proof. If $\gamma_n = 0$, then $I_n = 0$ and the estimate clearly holds. If $\gamma_n \neq 0$, then by (5.15) and the preceding lemma,

$$\left|4I_n/\gamma_n^2\right| \leqslant 9 \sup_{\lambda \in G_n} |\chi_n(\lambda)| \leqslant 4608(1 + \|\varphi\|_{H^1}^2). \quad \blacksquare$$

Proof of Theorem 23.4. Suppose $\varphi \in H_c^w \cap \tilde{\mathcal{W}}$ and choose $\langle N\rangle \geqslant 8\|\varphi\|_w^2 > N$. Then by the preceding proposition

$$\sum_{|n|\geqslant N} w_{2n}^2 |I_n| \leqslant 4608\|\varphi\|_{H^1}^2 \sum_{|n|>N} w_{2n}^2 |\gamma_n|^2,$$

and the gap lengths may be estimated by Proposition 25.5

$$\sum_{|n|\geqslant N} w_{2n}^2 |\gamma_n|^2 \leqslant 6\|R_N\varphi\|_w^2 + 144\|\varphi\|_w^4 \leqslant 144(1 + \|\varphi\|_w^2)\|\varphi\|_w^2.$$

In particular, the mapping

$$H_c^w \cap \tilde{\mathcal{W}} \to [0,\infty), \qquad \varphi \mapsto \sum_{n\in\mathbb{Z}} w_{2n}^2 |I_n|,$$

is continuous. Suppose φ is of real type, then the remaining actions for $|n| < N$ may be estimated by

$$\sum_{|n|<N} w_{2n}^2 |I_n| \leqslant w_{2N-2}^2 \sum_{n\in\mathbb{Z}} I_n \leqslant w[16\|\varphi\|_w^2]^2 \|\varphi\|_{L^2}^2.$$

Since $w \in \mathcal{M}^1$ is growing with at least linear speed, we thus find on H_r^w

$$\sum_{n\in\mathbb{Z}} w_{2n}^2 |I_n| \leqslant 2^{20} w[16\|\varphi\|_w^2]^2 \|\varphi\|_w^2.$$

For any nonzero potential φ this estimate extends by continuity to a complex neighborhood of φ within H_c^w with just the absolute constant doubled. On the other hand, sufficiently close to the zero potential we have $\|\varphi\|_w^2 < 1/8$. In this case we may choose $N = 0$ such that

$$\sum_{n\in\mathbb{Z}} w_{2n}^2 |I_n| \leqslant 2^{20}(1 + \|\varphi\|_w)^4 \|\varphi\|_{H^1}^2.$$

Consequently, on some sufficiently small open neighborhood of H_r^w in H_c^w,

$$\|I(\varphi)\|_{\ell_{w,1}} \leqslant 2^{21} w[16\|\varphi\|_w^2]^2 \|\varphi\|_w^2. \quad \blacksquare$$

Appendices

A. Periodic Distributions and Fourier Lebesgue Spaces

In this appendix we collect some basic facts about periodic distributions – see e.g. [41, 64] for further details.

We denote by $S(\mathbb{T}_a)$ the vector space of all smooth complex-valued functions of positive period a,

$$S(\mathbb{T}_a) \coloneqq \{f \in C^\infty(\mathbb{R}) : f(x+a) = f(x) \quad \forall\, x \in \mathbb{R}\},$$

which is endowed with the topology generated by the sequence of norms

$$\|f\|_{C^N} = \sum_{0 \leqslant n \leqslant N} \|D_x^n f\|_{L^\infty}, \qquad N \geqslant 0.$$

Here $D_x^n f$ denotes the classical nth derivative of f with respect to x. This vector space is a Fréchet space which is dense in the Banach space $C(\mathbb{T}_a)$ consisting of all continuous functions $f\colon \mathbb{R} \to \mathbb{C}$ of period a endowed with the sup-norm. Indeed, the functions $\mathrm{e}_{2m/a}(x) \coloneqq \mathrm{e}^{\mathrm{i}2m\pi x/a}$, $m \in \mathbb{Z}$, generate an algebra which is dense in $C(\mathbb{T}_a)$.

A *periodic distribution* u *of period* a is a continuous linear functional on the Fréchet space $S(\mathbb{T}_a)$. We denote the *action* of u on any element $f \in S(\mathbb{T}_a)$ by $[u, f]$. In this context, the elements of $S(\mathbb{T}_a)$ are also called *test functions*. Note that any linear map u on $S(\mathbb{T}_a)$ is continuous and hence a distribution if and only if there exists an integer $N \geqslant 0$ and a constant $C_N > 0$ so that

$$\lvert [u, f] \rvert \leqslant C_N \|f\|_{C^N}, \qquad f \in S(\mathbb{T}_a). \tag{A.1}$$

The space of all periodic distributions of period a is denoted by $S'(\mathbb{T}_a)$. The *order* of the distribution u is the least possible integer $N \geqslant 0$ so that (A.1) holds.

An element $u \in S'(\mathbb{T}_a)$ admits a continuous extension u_0 to the Banach space $C(\mathbb{T}_a)$ if and only if u has order zero. In this case, the extension is unique and by the Riesz representation theorem there exists a uniquely determined finite complex valued Radon measure μ so that

$$[u_0, f] = \frac{1}{a} \int_0^a f(x)\, \mathrm{d}\mu(x). \tag{A.2}$$

On the other hand, any finite complex Borel measure on $\mathbb{T}_a$ is inner regular and hence a Radon measure defining a periodic distribution in view of (A.2). Therefore, we may identify the space $C'(\mathbb{T}_a)$ of periodic distributions of order zero with the space $\mathcal{M}(\mathbb{T}_a)$ of finite complex Borel measures on $\mathbb{T}_a$. In particular, any element $u \in L^1(\mathbb{T}_a)$ defines a periodic distribution of order zero by

$$[u, f] \coloneqq \langle u, \overline{f} \rangle_{\mathbb{T}_a} \coloneqq \frac{1}{a} \int_0^a u(x) f(x)\; \mathrm{d}x.$$

To any periodic distribution $u \in S'(\mathbb{T}_a)$ we associate its sequence of *Fourier coefficients* $(u_m)_{m\in\mathbb{Z}}$,

$$u_m := [u, \mathrm{e}_{-2m/a}], \qquad m \in \mathbb{Z},$$

which is a natural extension of the Fourier coefficients of any $L^1(\mathbb{T}_a)$ function. As it turns out, any distribution $u \in S'(\mathbb{T}_a)$ is uniquely determined by its Fourier coefficients.

Theorem A.1 *(i) If u is a periodic distribution of order N, then*

$$|u_m| = O(|m|^N), \qquad |m| \to \infty.$$

(ii) Conversely, for any sequence $(\alpha_m) \subset \mathbb{C}$ with the property that $\alpha_m = O(|m|^N)$ for some $N \geqslant 0$ there exists a uniquely determined periodic distribution $u \in S'(\mathbb{T}_a)$ so that

$$u_m = \alpha_m. \quad \rtimes$$

Proof. See e.g. [64, Exercise 7.22]. ∎

The derivative $\partial_x u$ of a periodic distribution u is defined by $[\partial_x u, f] := -[u, D_x f]$. In particular, the Fourier coefficients of the derivative $\partial_x u$ satisfy

$$(\partial_x u)_m = \mathrm{i}(2m\pi/a)u_m, \qquad m \in \mathbb{Z}.$$

This motivates to introduce for any $s \geqslant 0$ and any $1 \leqslant p \leqslant \infty$ the *Fourier Lebesgue space*

$$FL^{s,p}(\mathbb{T}_a) := \{u \in S'(\mathbb{T}_a) : (u_m)_{m\in\mathbb{Z}} \in \ell^{s,p}_{\mathbb{C}}\}.$$

This space is a Banach space when endowed with the norm

$$\|u\|_{FL^{s,p}(\mathbb{T}_a)} := \|(u_m)_{m\in\mathbb{Z}}\|_{\ell^{s,p}},$$

which is reflexive for $1 < p < \infty$. To simplify notation we denote $FL^p(\mathbb{T}_a) \equiv FL^{0,p}(\mathbb{T}_a)$.

For any element $u \in FL^{s,p}(\mathbb{T}_a)$, the *Fourier series*

$$\sum_{m\in\mathbb{Z}} u_m \mathrm{e}_{2m/a}$$

converges in the $FL^{s,p}$ norm to a periodic distribution with Fourier coefficients u_m and hence in view of Theorem A.1 to the element u.

For $p = 2$ we obtain the well-known Sobolev spaces $H^s(\mathbb{T}_a) = FL^{s,2}(\mathbb{T}_a)$. In particular, we have $FL^2(\mathbb{T}_a) = L^2(\mathbb{T}_a)$ and by Parseval's identity for any $u \in FL^2(\mathbb{T}_a)$

$$\|u\|_{FL^2(\mathbb{T}_a)} = \|u\|_{L^2(\mathbb{T}_a)}.$$

However, for any other $p \neq 2$ the Fourier Lebesgue space $FL^p(\mathbb{T}_a)$ is quite different from any $L^q(\mathbb{T}_a)$ as we will see momentarily. First note that for $u \in L^1(\mathbb{T}_a)$ we have the estimate

$$|u_m| \leqslant \|u\|_{L^1(\mathbb{T}_a)},$$

hence $\|u\|_{FL^\infty(\mathbb{T}_a)} \leqslant \|u\|_{L^1(\mathbb{T}_a)}$. By the Riesz-Thorin Interpolation Theorem we obtain the following result.

Lemma A.2 *For any* $2 \leqslant p \leqslant \infty$ *the embedding* $L^{p'}(\mathbb{T}_a) \subset FL^p(\mathbb{T}_a)$, *with* $1/p + 1/p' = 1$, *is continuous.* ⋊

The space $FL^p(\mathbb{T}_a)$ turns out to be much bigger than $L^{p'}(\mathbb{T}_a)$ if $2 < p \leqslant \infty$. Indeed, we have the following result – see [41, Section IV, Remark 2.4].

Lemma A.3 *There exists an element in* $\bigcap_{p>2} FL^p(\mathbb{T}_a)$ *which is not a measure.* ⋊

Conversely, if one assumes that u is a periodic distribution with Fourier coefficients $(u_m)_{m\in\mathbb{Z}} \in \ell^1_{\mathbb{C}}$, then its Fourier series converges absolutely to a uniformly continuous function u, and one has the estimate

$$\|u\|_{C^0(\mathbb{T}_a)} \leqslant \|u\|_{FL^1(\mathbb{T}_a)}.$$

The space $FL^1(\mathbb{T}_a)$ turns out to be an algebra – the so called Wiener algebra $\mathbb{A}(\mathbb{T}_a)$ – however, we will not make use of this fact. The dual $FL^\infty(\mathbb{T}_a)$ of the space $FL^1(\mathbb{T}_a)$ is the space of *pseudo measures.*

Using interpolation one obtains the following embedding converse to Lemma A.2.

Lemma A.4 *For any* $1 \leqslant p \leqslant 2$ *the embedding* $FL^p(\mathbb{T}_a) \subset L^{p'}(\mathbb{T}_a)$ *is continuous.* ⋊

This time, the space $L^{p'}(\mathbb{T}_a)$ turns out to be much larger than $FL^p(\mathbb{T}_a)$.

Lemma A.5 *There exists a continuous function* $u \in C^0(\mathbb{T}_a)$ *which is not contained in* $\bigcup_{1\leqslant p<2} FL^p(\mathbb{T}_a)$. ⋊

Proof. One checks that $u(x) = \sum_{n\geqslant 2} \frac{\mathrm{e}^{\mathrm{i}n\log n}}{n^{1/2}(\log n)^2}\mathrm{e}^{\mathrm{i}n\pi x}$ is such an element – see e.g. [41, Section IV, Remark 2.3] and the references therein for further details. ∎

Remark A.6. By the Riemann-Lebesgue Lemma, the Fourier coefficients of any function $u \in L^1(\mathbb{T}_a)$ converge to zero. However, it is an immediate consequence of the open mapping theorem that the Fourier transform

$$F\colon L^1(\mathbb{T}_a) \to c_0$$

is not onto. In fact, a useful characterization of $L^1(\mathbb{T}_a)$ functions in terms of their Fourier coefficients still eludes our knowledge. ⊸

The product of two periodic distributions u and v is in view of Theorem A.1 well defined if and only if

$$w_m = \sum_{l\in\mathbb{Z}} u_{m-l}v_l$$

is convergent for any $m \in \mathbb{Z}$ and w_m grows at most polynomially in m. By Young's inequality for the convolution of sequences – see Lemma B.2 – we obtain the following estimate for products of elements of the Fourier Lebesgue spaces.

Lemma A.7 (Young-inequality) *Suppose* $1 \leqslant r, p, q \leqslant \infty$ *with* $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *and* $u \in FL^p(\mathbb{T}_a)$, $v \in FL^q(\mathbb{T}_a)$. *Then* uv *is an element of* $FL^r(\mathbb{T}_a)$ *satisfying the estimate*

$$\|uv\|_{FL^r(\mathbb{T}_a)} \leqslant \|u\|_{FL^p(\mathbb{T}_a)} \|v\|_{FL^q(\mathbb{T}_a)}. \qquad \rtimes \tag{A.3}$$

We conclude this section by noting the following two Sobolev embedding results. The first results shows in particular, that any element of $FL^{1,p}(\mathbb{T}_a)$, $1 \leqslant p < \infty$, is actually a continuous function.

Lemma A.8 *For any* $1 \leqslant p < \infty$, *the embedding* $FL^{1,p}(\mathbb{T}_a) \hookrightarrow FL^1(\mathbb{T}_a)$ *is continuous. In particular, for* $u \in FL^{1,p}(\mathbb{T}_a)$ *and* $v \in FL^p(\mathbb{T}_a)$, *the product* uv *is an element of* $FL^p(\mathbb{T}_a)$ *and we have the estimate*

$$\|uv\|_{FL^p(\mathbb{T}_a)} \leqslant \|u\|_{FL^{1,p}(\mathbb{T}_a)} \|v\|_{FL^p(\mathbb{T}_a)}. \qquad \rtimes \tag{A.4}$$

Proof. For any $u \in FL^{1,p}(\mathbb{T}_a)$ we have by Hölder's inequality

$$\|u\|_{FL^1(\mathbb{T}_a)} = \|(u_m)\|_{\ell^1} \leqslant \left\| \left(\frac{1}{\langle 2m/a \rangle} \right) \right\|_{\ell^{p'}} \|(\langle 2m/a \rangle u_m)\|_{\ell^p},$$

and $\left\| \left(\frac{1}{\langle 2m/a \rangle} \right) \right\|_{\ell^{p'}}$ is finite if $p' > 1$. ∎

Finally, we note that an arbitrary element of $H^s[0,a]$, $s \geqslant 0$, is not a continuous 1-periodic function and thus cannot be in $FL^1(\mathbb{T}_a)$. However, it is an element of $FL^q(\mathbb{T}_a)$ with q sufficiently large.

Lemma A.9 *For any* $0 \leqslant s < 1/2$

$$H^s[0,a] \hookrightarrow FL^q(\mathbb{T}_a), \qquad \forall\, q > \frac{1}{s + 1/2}.$$

In particular, $H^{1/2}[0,a] \hookrightarrow FL^q(\mathbb{T}_a)$ *for any* $q > 1$. $\rtimes$

Proof. For any $0 \leqslant s < 1/2$, we have $H^s[0,a] = H^s(\mathbb{T}_a)$ hence by Hölder's inequality for any $u \in H^s[0,a]$

$$\left(\sum_{m \in \mathbb{Z}} |u_m|^q \right)^{1/q} \leqslant \left(\sum_{m \in \mathbb{Z}} \frac{1}{\langle 2m/a \rangle^{sr}} \right)^{1/r} \left(\sum_{m \in \mathbb{Z}} \langle 2m/a \rangle^{2s} |u_m|^2 \right)^{1/2}$$

where $1/q = 1/r + 1/2$. The right hand side is finite if

$$sr > 1 \Leftrightarrow q > \frac{1}{s + 1/2}. \qquad ∎$$

B. Inequalities

In this appendix we collect three inequalities which are frequently used in this text. Throughout we denote by p' the Lebesgue exponent conjugated to p with $1 \leqslant p \leqslant \infty$.

Lemma B.1 *For any* $1 < p < \infty$ *and* $\alpha \geqslant 0$

$$\sum_{m \geqslant 1} \frac{1}{(\alpha + m)^{p'}} \leqslant \frac{p' + \alpha}{p' - 1} \frac{1}{(1 + \alpha)^{p'}} \leqslant \frac{p}{(1 + \alpha)^{p'-1}}. \qquad \rtimes$$

Proof. The claim follows immediately from the integral estimate of the sum. ∎

The Young inequality for the convolution of sequences can be stated in the following way.

Lemma B.2 (Young's inequality I) *Suppose* $1 \leqslant r, p_1, p_2 \leqslant \infty$ *with* $1 + \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, *and* $a \in \ell_{\mathbb{C}}^{p_1}$ *and* $b \in \ell_{\mathbb{C}}^{p_2}$, *then the convolution* $a * b$ *is an element of* $\ell_{\mathbb{C}}^{r}$ *satisfying the estimate*

$$\|a * b\|_{\ell^r} \leqslant \|a\|_{\ell^{p_1}} \|b\|_{\ell^{p_2}}. \quad \rtimes \tag{B.1}$$

We also note the following version of Young's inequality (B.2) for the convolution of three sequences.

Lemma B.3 (Young's inequality II) *Suppose*

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}, \qquad \gamma \geqslant p_1 \geqslant \beta \geqslant 0, \qquad p_2, p_3 \geqslant \alpha \geqslant 0,$$

and $a \in \ell_{\mathbb{C}}^{p_1}$, $b \in \ell_{\mathbb{C}}^{p_2}$, *and* $c \in \ell_{\mathbb{C}}^{p_3}$. *Then the convolution* $a * b * c$ *satisfies the estimate*

$$\left(\sum_k \left(\sum_l \left(\sum_m |a_{k-l} b_{l-m} c_m|^{\alpha} \right)^{\beta/\alpha} \right)^{\gamma/\beta} \right)^{1/\gamma} \leqslant \|a\|_{p_1} \|b\|_{p_2} \|c\|_{p_3}. \quad \rtimes$$

Proof. Let $d_l = \left(\sum_m |b_{l-m} c_m|^{\alpha}\right)^{1/\alpha}$ then the left hand side equals

$$\sum_k \left(\sum_l |a_{k-l}|^{\beta} |d_l|^{\beta} \right)^{\gamma/\beta} = \|a^{\beta} * d^{\beta}\|_{\gamma/\beta}^{\gamma/\beta} \leqslant \|a^{\beta}\|_{p_1/\beta}^{\gamma/\beta} \|d^{\beta}\|_{q_1/\beta}^{\gamma/\beta} = \|a\|_{p_1}^{\gamma} \|d\|_{q_1}^{\gamma}$$

provided that $p_1, q_1, \gamma \geqslant \beta$ and

$$\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{p_1} + \frac{1}{q_1}. \tag{B.2}$$

Note that given $p_1 \geqslant \beta$ and (B.2) the condition $q_1 \geqslant \beta$ is equivalent to $\gamma \geqslant p_1$. Furthermore,

$$\|d\|_{q_1}^{\gamma} = \|b^{\alpha} * c^{\alpha}\|_{q_1/\alpha}^{\gamma/\alpha} \leqslant \|b^{\alpha}\|_{p_2/\alpha}^{\gamma/\alpha} \|c^{\alpha}\|_{p_3/\alpha}^{\gamma/\alpha} = \|b\|_{p_2}^{\gamma} \|c\|_{p_3}^{\gamma},$$

provided that $q_1, p_2, p_3 \geqslant \alpha$ and

$$\frac{1}{\alpha} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{p_3}. \tag{B.3}$$

Note that (B.3) together with $p_2, p_3 \geqslant \alpha$ already implies that $q_1 \geqslant \alpha$. Moreover, (B.2) and (B.3) together yield

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}. \quad \blacksquare$$

C. Discrete Hilbert Transform on ℓ^p

In this appendix we recall some well known facts on the discrete Hilbert Transform on $\ell_{\mathbb{C}}^{p}$ – see e.g. [67].

Lemma C.1 *For any* $1 < p < \infty$, *the discrete Hilbert transform*

$$H \colon \ell_{\mathbb{C}}^{p} \to \ell_{\mathbb{C}}^{p}, \qquad (Hx)_n = \sum_{m \neq n} \frac{x_m}{m - n},$$

defines an isomorphism on $\ell_{\mathbb{C}}^{p}$. $\rtimes$

To simplify notation we define $\sigma^0 = (n\pi)_{n \in \mathbb{Z}}$. The following bound of the modified Hilbert transform is used in the next section.

Lemma C.2 *Suppose $\sigma = \sigma^0 + \tilde\sigma$ and $\rho = \sigma^0 + \tilde\rho$ are sequences of complex numbers with $\tilde\sigma, \tilde\rho \in \ell^\infty_{\mathbb{C}}$ such that for some $c > 0$*

$$|\rho_m - \sigma_n| \geqslant c^{-1}|m-n|, \qquad m \neq n.$$

Then for any $1 < p < \infty$,

$$(Ax)_n = \pi \sum_{m\neq n} \frac{x_m}{\rho_m - \sigma_n}$$

defines a bounded linear operator A on $\ell^p_{\mathbb{C}}$ whose bound depends only on $\|\tilde\sigma\|_\infty$, $\|\tilde\rho\|_\infty$, c, and p. ⋊

Proof. To apply Lemma C.1, we write A in the form

$$\begin{aligned}(Ax)_n &= \sum_{m\neq n} \frac{x_m}{m-n}\left(1 + \frac{\sigma_n - n\pi}{\rho_m - \sigma_n} - \frac{\rho_m - m\pi}{\rho_m - \sigma_n}\right)\\ &= (Hx)_n + (F_\rho x)_n - (G^\sigma x)_n,\end{aligned}$$

where the operators F_ρ and G^σ are given by

$$(F_\sigma x)_n = \sum_{m\neq n} \frac{x_m}{m-n}\frac{\tilde\sigma_n}{\rho_m - \sigma_n}, \qquad (G^\rho x)_n = \sum_{m\neq n} \frac{x_m}{m-n}\frac{\tilde\rho_m}{\rho_m - \sigma_n}.$$

By assumption $|\rho_m - \sigma_n| \geqslant c^{-1}|m-n|$ for $m \neq n$ so that

$$\sum_{n\in\mathbb{Z}} |(F_\sigma x)_n|^p \leqslant c^p \|\tilde\sigma\|_\infty^p \sum_{n\in\mathbb{Z}} \left|\sum_{m\neq n} \frac{|x_m|}{|m-n|^2}\right|^p \leqslant C_p \|\tilde\sigma\|_\infty^p \|x\|_p^p,$$

where the second estimate follows from Young's inequality (B.1). A similar bound can be obtained for $\sum_{n\in\mathbb{Z}}|(G^\rho x)_n|^p$ which proves the claim. ∎

D. Infinite products

Let us first recall some definitions and facts on infinite products form [23]. Let $a := (a_n)_{n\geqslant 1}$ be a sequence of complex numbers. We say that the infinite product $\prod_{n\geqslant 1}(1 + a_n)$ *converges* if the limit $\lim_{N\to\infty} \prod_{1\leqslant n\leqslant N}(1 + a_n)$ exists, and $\prod_{n\geqslant 1}(1 + a_n)$ is said to be *absolutely convergent* if $\prod_{n\geqslant 1}(1 + |a_n|)$ converges. One verifies that absolute convergence implies convergence. A sufficient condition for absolute convergence is that $\|a\|_{\ell^1} := \sum_{n\geqslant 1}|a_n| < \infty$. The following inequality can be found in [23, Lemma C1].

Lemma D.1 *For any ℓ^1-sequence of complex numbers a_m with $|a_m| \leqslant 1/2$,*

$$\left|\prod_{m\in\mathbb{Z}} (1 + a_m) - 1\right| \leqslant A\mathrm{e}^S + B\mathrm{e}^{S+S^2}$$

with $A = \left|\sum_{m\in\mathbb{Z}} a_m\right|$, $B = \sum_{m\in\mathbb{Z}}|a_m|^2$, and $S = \sum_{m\in\mathbb{Z}}|a_m|$. ⋊

Remark D.2. We also have the following weaker bound just in terms of S,

$$\left|\prod_{m\in\mathbb{Z}} (1 + a_m) - 1\right| \leqslant \mathrm{e}^S - 1 \leqslant S\mathrm{e}^S,$$

however we need the more detailed estimate of Lemma D.1 in the sequel. ⊸

Remark D.3. By the same arguments as in the proof of [23, Lemma C1] one obtains the slightly refined estimate

$$\left| \prod_{m\in\mathbb{Z}} (1+a_m) - 1 - A \right| \leqslant \frac{|A|^2}{2} \mathrm{e}^S + B\mathrm{e}^{S+S^2}. \quad \multimap$$

We are mainly concerned with infinite products related to perturbations of the sine function

$$\sin\lambda = -\prod_{m\in\mathbb{Z}} \frac{m\pi - \lambda}{\pi_m},$$

where we replace the sequence $(m\pi)_{m\in\mathbb{Z}}$ by $\sigma = (\sigma_m)_{m\in\mathbb{Z}}$ with $\sigma = \sigma^0 + \tilde{\sigma}$ and $\tilde{\sigma}$ is an element of $\ell^p_{\mathbb{C}}$ for some $1 \leqslant p < \infty$. Such a sequence σ is called *simple* if all its elements are mutually distinct. In this case one has

$$\inf_{m\neq n} |\sigma_m - \sigma_n| > 0.$$

In the sequel, we adapt the presentation of [23, Appendix C] where the case $p = 2$ is considered.

Lemma D.4 *For $\sigma = \sigma^0 + \tilde{\sigma}$ with $\tilde{\sigma} \in \ell^p_{\mathbb{C}}$, $1 \leqslant p < \infty$ and $n \in \mathbb{Z}$,*

$$f_n(\lambda) = -\prod_{m\neq n} \frac{\sigma_m - \lambda}{\pi_m}$$

defines an analytic function on $\mathbb{C}\times\ell^p_{\mathbb{C}}$ with roots σ_m, $m \neq n$, listed with their multiplicities. In particular, if σ is simple, then f_n has simple roots σ_m, $m\neq n$, and

$$f_n^{-1}(\lambda) = -\prod_{m\neq n} \frac{\pi_m}{\sigma_m - \lambda}$$

is meromorphic with simple poles σ_m, $m \neq n$. $\rtimes$

Proof. To simplify matter we only consider the case $n = 0$. Define

$$a_m = \frac{\sigma_m - \lambda}{\pi_m} - 1 = \frac{\tilde{\sigma}_m - \lambda}{\pi_m}, \qquad m \neq 0,$$

then

$$a_m + a_{-m} = \frac{\tilde{\sigma}_m - \tilde{\sigma}_{-m}}{\pi_m} = \frac{\ell^p_m}{m} = \ell^1_m,$$

for $1 \leqslant p < \infty$ and

$$a_m a_{-m} = \frac{(\tilde{\sigma}_m - \lambda)(\lambda - \tilde{\sigma}_{-m})}{\pi_m^2} = \frac{\ell^\infty_m}{m^2} = \ell^1_m.$$

Therefore, the product $f_n(\lambda) = -\prod_{m\neq n}(1 + a_m)$ is locally uniformly convergent on $\mathbb{C}\times\ell^p_c$. Since every finite product is a polynomial in λ and σ_m, the analyticity of f_n follows from the locally uniform convergence and [23, Theorem A.4]. ∎

We recall the definition of the complex discs

$$D_n = \{\lambda \in \mathbb{C} : |\lambda - n\pi| < \pi/4\}.$$

Lemma D.5 *For $\sigma = \sigma^0 + \tilde{\sigma}$ with $\tilde{\sigma} \in \ell^p_{\mathbb{C}}$, $1 \leqslant p < \infty$,*

$$f(\lambda) = -\prod_{m\in\mathbb{Z}} \frac{\sigma_m - \lambda}{\pi_m}$$

defines an analytic function on $\mathbb{C} \times \ell^p_{\mathbb{C}}$ with roots σ_m, $m \in \mathbb{Z}$, which satisfies

$$f(\lambda) = (1 + o(1)) \sin \lambda$$

locally uniformly in σ and uniformly on $\Pi = \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} D_n$ as $|\lambda| \to \infty$. In more detail, for every $\varepsilon > 0$ there exists $\Lambda_0 > 0$ such that

$$\sup_{\lambda\in\Pi,\, |\lambda|>\Lambda_0} \left| \frac{f(\lambda)}{\sin\lambda} - 1 \right| < \varepsilon.$$

Here, Λ_0 can be chosen locally uniformly in σ. ⋊

Proof. By Lemma D.4, f is an entire function with roots σ_m, $m \in \mathbb{Z}$. Moreover, the quotient

$$\frac{f(\lambda)}{\sin\lambda} = \prod_{m\in\mathbb{Z}} \frac{\sigma_m - \lambda}{m\pi - \lambda},$$

is well defined and analytic on Π. Fix any $\lambda \in \Pi$, then there exists $n \in \mathbb{Z}$ such that $|\Re\lambda - n\pi| \leqslant \pi/2$ and $|\lambda - n\pi| > \pi/4$. Consequently,

$$\left| \frac{\sigma_m - \lambda}{m\pi - \lambda} - 1 \right| = \left| \frac{\tilde{\sigma}_m}{m\pi - \lambda} \right| \leqslant \frac{4|\tilde{\sigma}_m|}{1 + |n-m| + |\Im\lambda|}.$$

Let $S_n = \sum_{m\in\mathbb{Z}} \frac{|\tilde{\sigma}_m|}{1+|n-m|+|\Im\lambda|}$, then in view of Lemma B.1 there exists an absolute constant $c_p > 0$ so that

$$\begin{aligned} S_n &\leqslant \sum_{|m-n|\geqslant|n|/2} \frac{|\tilde{\sigma}_m|}{1 + |n-m| + |\Im\lambda|} + \sum_{|m-n|<|n|/2} \frac{|\tilde{\sigma}_m|}{1 + |n-m| + |\Im\lambda|} \\ &\leqslant c_p \left(\frac{\|\tilde{\sigma}\|_{\ell^p}}{1 + |n|^{1/p} + |\Im\lambda|^{1/p}} + \frac{\|R_n\tilde{\sigma}\|_{\ell^p}}{1 + |\Im\lambda|^{1/p}} \right). \end{aligned}$$

The right hand side tends to zero locally uniformly in σ and uniformly on Π as $|\lambda| \to \infty$, so the claim follows with Lemma D.1. ∎

Lemma D.6 *Suppose $\sigma = \sigma^0 + \tilde{\sigma}$ and $\rho = \rho^0 + \tilde{\rho}$ with ρ simple and $\tilde{\rho}, \tilde{\sigma} \in \ell^\infty_{\mathbb{C}}$. If $\tilde{\rho} - \tilde{\sigma} \in \ell^p_{\mathbb{C}}$, $1 < p < \infty$, and there exists $c > 0$ and $N \geqslant 0$ such that*

$$\min_{\lambda\in D_n} |\rho_m - \lambda| \geqslant c^{-1}|m-n|, \qquad m \neq n, \quad |n| \geqslant N,$$

then

$$\prod_{m\neq n} \frac{\sigma_m - \lambda}{\rho_m - \lambda}\bigg|_{D_n} = 1 + \ell^p_n,$$

uniformly in $\|\tilde{\rho}\|_\infty$ and $\|\sigma - \rho\|_p$. In more detail, choose $N_1 \geqslant N$ so that $\frac{2c}{N_1}\|\sigma - \rho\|_p + \|R_{N_1/2}(\sigma - \rho)\|_p \leqslant 1/2$, then

$$\sum_{|n|\geqslant N_1} \sup_{\lambda\in D_n} \left| \prod_{m\neq n} \frac{\sigma_m - \lambda}{\rho_m - \lambda} - 1 \right|^p \leqslant C_{c,p,\|\tilde{\rho}\|_\infty,\|\rho-\sigma\|_p} \|\rho - \sigma\|_p.$$ ⋊

Proof. Let $\alpha_m = \sigma_m - \rho_m$ and put

$$a_{m,n} = \left|\frac{\sigma_m - \lambda}{\rho_m - \lambda} - 1\right|_{D_n} = \left|\frac{\alpha_m}{\rho_m - \lambda}\right|_{D_n}.$$

Since $\alpha_m = \ell^p_m$ and $\min_{\lambda \in D_n}|\rho_m - \lambda| \geqslant c^{-1}|m-n|$ for $m \neq n$ and $|n| \geqslant N$, there exists $N_1 \geqslant N$ so that for all $m \in \mathbb{Z}$ and $|n| \geqslant N_1$

$$|a_{m,n}| \leqslant \begin{cases} \frac{2c}{n}\|\alpha\|_p \leqslant 1/2, & |m-n| \geqslant |n|/2, \\ c\|R_{n/2}\alpha\|_p \leqslant 1/2, & 1 \leqslant |m-n| \leqslant |n|/2. \end{cases}$$

Therefore, Lemma D.1 applies yielding

$$\sup_{\lambda \in D_n}\left|\prod_{m \neq n}\frac{\sigma_m - \lambda}{\rho_m - \lambda} - 1\right| = A_n \mathrm{e}^{S_n} + B_n \mathrm{e}^{S_n + S_n^2},$$

where $S_n = \sum_{m \neq n}\left|\frac{\alpha_m}{\rho_m - \lambda}\right|_{D_n}$, $A_n = \left|\sum_{m \neq n}\frac{\alpha_m}{\rho_m - \lambda}\right|_{D_n}$, and $B_n = \sum_{m \neq n}\left|\frac{\alpha_m}{\rho_m - \lambda}\right|^2_{D_n}$. By Hölder's inequality for any $p < \infty$

$$S_n \leqslant c\left(\sum_{m \neq n}\frac{1}{|m-n|^{p'}}\right)^{1/p'}\|\alpha\|_p \leqslant C_p\|\alpha\|_p.$$

Since ρ is simple, after possibly modifying finitely many D_n, $|n| < N$, we have

$$|\rho_m - \lambda| \geqslant \tilde{c}^{-1}|m-n|, \qquad m \neq n,$$

for all $n \in \mathbb{Z}$, hence $\sum_{n \in \mathbb{Z}}|A_n|^p \leqslant C_{p,\|\tilde{\sigma}\|_\infty,\|\tilde{\rho}\|_\infty}\|\alpha\|_p^p$ by Lemma C.2. Finally, if $p \geqslant 2$, then by Young's inequality

$$\sum_{|n| \geqslant N_1} B_n^{p/2} \leqslant c^p\left(\sum_{m \neq 0}\frac{1}{|m|^2}\right)^{p/2}\|\alpha\|_p^p \leqslant C_q\|\alpha\|_p^p$$

while if $1 < p \leqslant 2$ one can apply the inequality $(|a| + |b|)^{p/2} \leqslant |a|^{p/2} + |b|^{p/2}$ to obtain,

$$\sum_{|n| \geqslant N_1} B_n^{p/2} \leqslant c^2\left(\sum_{m \neq 0}\frac{1}{|m|^p}\right)\|\alpha\|_q^p \leqslant C_q\|\alpha\|_p^p.$$

Altogether we thus have

$$\sum_{|n| \geqslant N_1}\sup_{\lambda \in D_n}\left|\prod_{m \neq n}\frac{\sigma_m - \lambda}{\rho_m - \lambda} - 1\right|^p \leqslant C_{p,\|\tilde{\rho}\|_\infty,\|\alpha\|_p}\|\alpha\|_p^p. \quad \blacksquare$$

Remark D.7. In view of Remark D.3 one obtains under the same assumptions as in Lemma D.6 that

$$\sup_{\lambda \in D_n}\left|\prod_{m \neq n}\frac{\sigma_m - \lambda}{\rho_m - \lambda} - 1 - \sum_{m \neq n}\frac{\sigma_m - \rho_m}{\rho_m - \lambda}\right| = \ell_n^{p/2} + \ell_n^{1+}. \quad \multimap$$

As an immediate consequence of Lemma D.6 and the product representation of the sine, we obtain the following result.

Lemma D.8 *Let $\sigma = \sigma^0 + \tilde{\sigma}$ with $\tilde{\sigma} \in \ell^p_{\mathbb{C}}$, $1 < p < \infty$, and $n \in \mathbb{Z}$,*

$$\frac{1}{\pi_n}\prod_{m \neq n}\frac{\sigma_m - \lambda}{\pi_m} = \frac{\sin\lambda}{\lambda - n\pi}(1 + \ell^p_n) = \frac{\sin\lambda}{\lambda - n\pi} + \ell^p_n, \qquad \lambda \in D_n,$$

locally uniformly in $\tilde{\sigma}$. $\rtimes$

We conclude this appendix with the following estimate.

Lemma D.9 *Let $\sigma = \sigma^0 + \tilde{\sigma}$ with $\tilde{\sigma} \in \ell^p_{\mathbb{C}}$, $1 < p < \infty$,*

$$f(\lambda) = -\prod_{m\in\mathbb{Z}} \frac{\sigma_m - \lambda}{\pi_m}.$$

Then for any sequence (λ_n) with $\lambda_n \in D_n$,

$$f(\lambda_n) = \sin(\lambda_n) + \ell^p_n,$$

locally uniformly in σ. ⋊

Proof. Let

$$f_n(\lambda) = -\prod_{m\neq n} \frac{\sigma_m - \lambda}{\pi_m}, \qquad s_n(\lambda) = -\prod_{m\neq n} \frac{m\pi - \lambda}{\pi_m},$$

and note that for any sequence λ_n with $\lambda_n \in D_n$ by Lemma D.6

$$\frac{f_n(\lambda_n)}{s_n(\lambda_n)} = 1 + \ell^p_n.$$

Therefore,

$$\begin{aligned} f(\lambda_n) - \sin(\lambda_n) &= \frac{\sigma_n - \lambda_n}{\pi_n} f_n(\lambda_n) - \frac{n\pi - \lambda_n}{\pi_n} s_n(\lambda_n) \\ &= \frac{s_n(\lambda_n)}{\pi_n}\Big[(\sigma_n - \lambda_n)(1 + \ell^p_n) - (n\pi - \lambda_n)\Big] \\ &= \frac{s_n(\lambda_n)}{\pi_n}\Big[\tilde{\sigma}_n + (\sigma_n - \lambda_n)\ell^p_n\Big] = \ell^p_n, \end{aligned}$$

where we used that $|\sigma_n - \lambda_n| = O(1)$ and $\frac{s_n(\lambda_n)}{\pi_n} = O(1)$. ∎

E. Miscellaneous lemmas

The following is an extension of [23, Lemma F.4] to the case $p \neq 2$.

Lemma E.1 (Interpolation Lemma) *Suppose ϕ is an entire function with*

$$\sup_{\lambda\in C_n} \left|\frac{\phi(\lambda)}{\sin\lambda}\right| \to 0,$$

as $n \to \infty$ for the circles $C_n = \{|\lambda| = n\pi + \pi/2\}$. Then for any simple sequence $\sigma_n = n\pi + \ell^p_n$, $1 \leqslant p < \infty$, and any $z \in \mathbb{C}$ with $z \neq \sigma_m$, $m \in \mathbb{Z}$,

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi(\sigma_n) \prod_{m\in\mathbb{Z}} \frac{\sigma_m - z}{\sigma_m - \sigma_n}.$$ ⋊

Proof. By Lemma D.5

$$\psi(\lambda) = -\prod_{m\in\mathbb{Z}} \frac{\sigma_m - \lambda}{\pi_m}$$

defines an entire function whose roots are precisely the σ_m, $m \in \mathbb{Z}$, and which is of the form $(\sin\lambda)(1 + o(1))$ on the circles C_n. For any fixed $z \in \mathbb{C}$, the function

$$g(\lambda) = \frac{1}{\lambda - z}\frac{\phi(\lambda)}{\psi(\lambda)}$$

is thus a meromorphic with simple poles at z and all the σ_n, and satisfies

$$\sup_{\lambda\in C_n} |(\lambda - z)g(\lambda)| = \sup_{\lambda\in C_n} \left|\frac{\phi(\lambda)}{\sin\lambda}\right|(1 + o(1)) \to 0.$$

Applying the residue theorem to each circle C_n and letting $n \to \infty$ shows that the sum of all the residues of g is zero. Provided $z \neq \sigma_m$ for all $m \in \mathbb{Z}$ we thus obtain

$$\begin{aligned} 0 &= \operatorname{Res}_z g + \sum_{n\in\mathbb{Z}} \operatorname{Res}_{\sigma_n} g \\ &= \frac{\phi(z)}{\psi(z)} + \sum_{n\in\mathbb{Z}} \phi(\sigma_n) \frac{\pi_n}{\sigma_n - z} \prod_{m\neq n} \frac{\pi_m}{\sigma_m - \sigma_n}. \end{aligned}$$

Solving for $\phi(z)$ gives the claim. ∎

Lemma E.2 (Identity Theorem for real subspaces) *Let $X_{\mathbb{R}}$ be a real Banach space and denote by X its complexification. Suppose $U \subset X$ is an open connected neighborhood and $f: U \to \mathbb{C}$ is an analytic. If f vanishes on $U_{\mathbb{R}} = U \cap X_{\mathbb{R}}$, then $f \equiv 0$.* ⋊

Proof. Near any $u \in U_{\mathbb{R}}$ the map f is represented by its Taylor series,

$$f(u+h) = \sum_{n\geqslant 0} \frac{1}{n!} d_u^n f(h,\dots,h),$$

where the series converges absolutely and uniformly (cf. e.g. [23, Theorem A.3]). Since $f|_{U_{\mathbb{R}}} = 0$, it follows that for any $h \in X_r$ and any $n \geqslant 0$, $d_u^n f(h,\dots,h) = 0$. As f is analytic, $d_u^n f$ is symmetric and $\mathbb{C}$-multilinear, hence it follows from the polarization identity that $d_u^n f(h,\dots,h) = 0$ holds also for any h in the complexification X of $X_{\mathbb{R}}$. This implies $f \equiv 0$ in a neighborhood V_u of u with $V_u \subset U$. Since U is connected it follows that $f \equiv 0$ on all of U by the ordinary identity theorem. ∎

Lemma E.3 *Suppose $|\nu_n - n\pi| \leqslant \pi/4$ for $|n|$ sufficiently large, then*

$$\|e^{i\nu_n x}\|_{FL^q(\mathbb{T}_1)} = O(1), \qquad \forall\ q > 1.$$ ⋊

Proof. Since

$$\langle e^{i\nu_n x} - e^{in\pi x}, e^{2im\pi x}\rangle = O\left(\frac{1}{1+|n-2m|}\right),$$

uniformly in n and m we find $\|e_{\nu_n}\|_{FL^q(\mathbb{T}_1)} = O(1)$ for any $q > 1$. ∎

F. Analyticity of F

In this appendix we prove the analyticity of the primitive

$$F_n(\mu,\psi) = \frac{1}{2}\left(\int_{\lambda_n^-(\psi)}^{\mu} \omega(\lambda,\psi)\,d\lambda + \int_{\lambda_n^+(\psi)}^{\mu} \omega(\lambda,\psi)\,d\lambda\right), \qquad \omega = \frac{\Delta^\bullet}{\sqrt[c]{\Delta^2-4}},$$

introduced in Section 19. The proof is similar as in [59] with minor adaptions for the case $n \neq 0$ and $p \neq 2$. We first note the following observation on which the proof relies.

Lemma F.1 *(i) Suppose $\varphi \in \mathcal{W}^p$, $1 < p < \infty$, and $\gamma_n(\varphi) = 0$ for some $n \in \mathbb{Z}$, then there exists an open neighborhood V_n of φ such that*

$$|\lambda_n^\bullet - \tau_n| \leqslant |\gamma_n|/2, \qquad \psi \in V_n.$$

(ii) For each $\varphi \in FL_r^p$, $1 < p < \infty$, there exists an open neighborhood $V \subset FL_c^p$ such that

$$|\lambda_m^\bullet - \tau_m| \leqslant |\gamma_m|, \qquad m \in \mathbb{Z}, \quad \psi \in V.$$ ⋊

Proof. (i) For any potential in $V_n = V_\varphi$ we have

$$0 = \frac{1}{4}(\Delta^2 - 4)^\bullet\Big|_{\lambda_n^\bullet} = \Big(2(\tau_n - \lambda)\Delta_n - \Big((\tau_n - \lambda)^2 - \gamma_n^2/4\Big)\,\Delta_n^\bullet\Big)\Big|_{\lambda_n^\bullet},$$

where the function Δ_n is analytic on $\mathbb{C} \times \mathcal{W}^p$ and given by

$$\Delta_n(\lambda) \coloneqq \frac{1}{\pi_n^2}\prod_{m \neq n}\frac{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}{\pi_m^2}.$$

The zeroes of Δ_n are precisely the eigenvalues $\lambda_m^\pm$ for $m \neq n$, thus Δ_n does not vanish on $U_n \times V_n$ and we have $|\Delta_n| \geqslant s > 0$ and $|\Delta_n^\bullet| \leqslant r$ uniformly on $U_n \times V_n$. Since $\gamma_n(\varphi) = 0$, we may shrink V_n, if necessary, to the effect that

$$|\gamma_n(\psi)|r \leqslant s, \qquad \psi \in V_n.$$

To simplify notation put $f = 2(\tau_n - \lambda)\Delta_n$ and $g = (\Delta^2 - 4)^\bullet/4$. By Lemma 10.11 $\gamma_n(\psi) = 0$ implies $\lambda_n^\bullet(\psi) = \tau_n(\psi)$, hence we may assume $\gamma_n(\psi) \neq 0$. In this case, we find on the boundary of the disc $D_\psi = \{|\lambda - \tau_n(\psi)| \leqslant |\gamma_n(\psi)|/2\}$,

$$|f - g|_{\partial D_\psi} \leqslant |\gamma_n(\psi)|^2 r/2 < s|\gamma_n(\psi)| \leqslant |f|_{\partial D_\psi}.$$

Thus, by Rouché's Theorem, f and g have the same number of roots in D_ψ, and since $\lambda_n^\bullet$ is the only root of g contained in U_n, the claim (i) follows.

(ii) Recall from Lemma 10.10 that for each $\varphi \in \mathcal{W}^p$ there exists a neighborhood V and an $M \geqslant 0$ such that

$$\sum_{|m| \geqslant M} |\lambda_m^\bullet - \tau_m|^2/|\gamma_m|^4 \leqslant 1, \qquad \psi \in V.$$

After possibly increasing M we have $|\lambda_m^\bullet - \tau_m| \leqslant |\gamma_m|$ on V for all $|m| \geqslant M$. Suppose that, in addition, $\varphi \in FL_r^p$. If $\gamma_m(\varphi) = 0$ for some $|m| < M$, then we may shrink V according to (i) to obtain the desired inequality. On the other hand, if $\gamma_m(\varphi) \neq 0$, then

$$|\lambda_m^\bullet(\varphi) - \tau_m(\varphi)| \leqslant |\gamma_m(\varphi)|/2,$$

and since $\lambda_m^\bullet$, τ_m, and $|\gamma_m|$ are continuous on $\mathcal{W}^p$, the desired inequality follows after possibly shrinking V. ∎

Lemma F.2 *Let $\varphi \in \mathcal{W}^p$, $1 < p < \infty$, and $n \in \mathbb{Z}$. Then for any $\nu \in \partial U_n$,*

$$F_n(\nu) = \frac{1}{2}\left(\int_{\lambda_n^-}^{\nu} \omega\,\mathrm{d}\lambda + \int_{\lambda_n^+}^{\nu} \omega\,\mathrm{d}\lambda\right),$$

is analytic on V_φ. ⋊

Proof. We want to apply [23, Theorem A.6]. First note that for any $\mu, \nu \in \partial U_n$, $\int_\nu^\mu \omega$ is analytic on V_φ by Lemma 10.11. Hence at each step of the proof we might change ν at our convenience.

In a first step we prove that F_n is analytic on $V_\varphi \setminus Z_n$, where $Z_n = \{\psi \in \mathcal{W}^p : \gamma_n^2(\psi) = 0\}$. Note that γ_n^2 is analytic on $\mathcal{W}^p$ thus Z_n is an analytic subvariety of $\mathcal{W}^p$. Fix $\varphi_0 \in V_\varphi \setminus Z_n$. Recall that λ_n^-, λ_n^+ are lexicographically ordered and might therefore not even be continuous near φ_0. However, since the eigenvalues $\lambda_n^+(\varphi_0)$, $\lambda_n^-(\varphi_0)$ are both simple, there exist a neighborhood $V \subset V_\varphi \setminus Z_n$ of

φ_0 and two analytic functions $\rho_1, \rho_2 \colon V \to \mathbb{C}$ satisfying the set inequality $\{\rho_1, \rho_2\} = \{\lambda_n^-, \lambda_n^+\}$ on V and in addition

$$|\rho_1 - \lambda_n^-(p_0)| < \frac{1}{4} \min\left[\operatorname{dist}(\lambda_n^-(p_0), \partial U_n), |\gamma_n(p_0)|\right],$$
$$|\rho_2 - \lambda_n^+(p_0)| < \frac{1}{4} \min\left[\operatorname{dist}(\lambda_n^+(p_0), \partial U_n), |\gamma_n(p_0)|\right].$$

In particular, $\rho_1(\varphi_0) = \lambda_n^-(\varphi_0)$ and $\rho_2(\varphi_0) = \lambda_n^+(\varphi_0)$. Let ν_1, $\nu_2 \in \partial U_n$ be elements of the form $\nu_1 = \lambda_n^-(\varphi_0) - \sigma_1\gamma_n(\varphi_0)$, $\nu_2 = \lambda_n^+(\varphi_0) + \sigma_2\gamma_n(\varphi_0)$ with σ_1, $\sigma_2 > 0$. Then for each $\psi \in V$, the line segments $[\rho_1, \nu_1]$ and $[\rho_2, \nu_2]$ are admissible paths and

$$2F_n(\nu) = \int_{\rho_1}^{\nu_1} \omega + \int_{\nu_1}^{\nu} \omega + \int_{\rho_2}^{\nu_2} \omega + \int_{\nu_2}^{\nu} \omega.$$

As already noted above, $\int_{\nu_1}^{\nu} \omega$ and $\int_{\nu_2}^{\nu} \omega$ are both analytic on V_φ. We now prove that for $i = 1, 2$, $\int_{\rho_i}^{\nu_i} \omega$ is analytic on V. Since the cases $i = 1$ and $i = 2$ are treated in the same way we concentrate on $i = 1$. With the parametrization $\lambda_t = \rho_1 + t(\nu_1 - \rho_1)$ one gets

$$\int_{\rho_1}^{\nu_1} \omega = -\mathrm{i}(\nu_1 - \rho_1) \int_0^1 \frac{\lambda_n^\bullet - \lambda_t}{w_n(\lambda_t)} \chi_n(\lambda_t)\, \mathrm{d}t,$$

where $\chi_n(\lambda)$ defined in (4.7) is given by

$$\chi_n(\lambda) = \prod_{m \neq n} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)} \tag{F.1}$$

is analytic on $U_n \times V_\varphi$ – see Corollary 10.6. Taking into account that

$$(\tau_n - \lambda_t)^2 - \gamma_n^2/4 = (\rho_1 - \lambda_t)(\rho_2 - \lambda_t) = t(\rho_1 - \nu_1)(\rho_2 - \lambda_t),$$

one obtains

$$w_n(\lambda_t) = (\tau_n - \lambda_t) \sqrt[+]{t} \sqrt[+]{\frac{(\rho_1 - \nu_1)(\rho_2 - \lambda_t)}{(\tau_n - \lambda_t)^2}}.$$

Moreover, at φ_0 we have for any $0 \leqslant t \leqslant 1$,

$$\frac{(\rho_1 - \nu_1)(\rho_2 - \lambda_t)}{(\tau_n - \lambda_t)^2} = \frac{\sigma_1(1 + t\sigma_1)\gamma_n^2}{(1/2 + t\sigma_1)^2\gamma_n^2} = \frac{\sigma_1(1 + t\sigma_1)}{(1/2 + t\sigma_1)^2} \geqslant \frac{\sigma_1}{1/2 + \sigma_1} > 0.$$

Thus, after possibly shrinking V, the mappings $(t, p) \mapsto \sqrt[+]{\frac{(\rho_1 - \nu_1)(\rho_2 - \lambda_t)}{(\tau_n - \lambda_t)^2}}$ and $(t, p) \mapsto \tau_n - \lambda_t$ are continuous on $[0, 1] \times V$, analytic on V for every fixed $0 \leqslant t \leqslant 1$, and uniformly bounded away from zero by some $c_1 > 0$. Since χ_n is analytic on $U_n \times V_\varphi$, and ρ_1, ρ_2, and $\lambda_n^\bullet$ are analytic on $V \subset V_\varphi$, it then follows that

$$\int_{\rho_1}^{\nu_1} \omega = -\mathrm{i}(\nu_1 - \rho_1) \int_0^1 \frac{\chi_n(\lambda_t)}{\tau_n - \lambda_t} \frac{\lambda_n^\bullet - \lambda_t}{\sqrt[+]{\frac{(\rho_1 - \nu_1)(\rho_2 - \lambda_t)}{(\tau_n - \lambda_t)^2}}} \frac{\mathrm{d}t}{\sqrt[+]{t}},$$

is analytic on V as well.

In a second step we show that the restriction of F_n to $Z_n \cap V_\varphi$ is weakly analytic. Note that on $Z_n \cap V_\varphi$, F_n coincides with the function $-\mathrm{i} \int_{\tau_n}^{\nu} \chi_n(\lambda)\, \mathrm{d}\lambda$ with χ_n given by (4.7), where the path of integration is chosen in U_n but otherwise arbitrary. Thus $F_n|_{Z_n \cap V_\varphi}$ is weakly analytic.

In a third and final step we prove that F_n is continuous on V_φ. By the considerations above, F_n is continuous in each point of $V_\varphi \setminus Z_n$ and the restriction of F_n to $Z_n \cap V_\varphi$ is continuous. Hence it remains to show that for any $\varphi_\infty \in Z_n \cap V_\varphi$ and any sequence $(\varphi_k)_{k \geqslant 1} \subset V_\varphi \setminus Z_n$ with $\varphi_k \to \varphi_\infty$ in

V_φ it follows that $F_n(\varphi_k) \to F_n(\varphi_\infty)$ as $k \to \infty$. By Lemma 10.10 $\lambda_n^\bullet - \tau_n = O(\gamma_n^2)$ locally uniformly around φ_∞. Since $\gamma_n(\varphi_k) \to 0$ as $k \to \infty$, there exists $k_0 \geqslant 1$ so that

$$|\lambda_n^\bullet(\varphi_k) - \tau_n(\varphi_k)| \leqslant |\gamma_n(\varphi_k)|/2, \qquad k \geqslant k_0. \tag{F.2}$$

Without loss of generality we may assume that $k_0 = 1$. For each φ_k consider the parametrization of $[\lambda_n^-, \tau_n]$, given by $[0,1] \to G_n$, $t \mapsto \lambda_t = \tau_n - t\gamma_n/2$. Then $w_n(\lambda_t)^2 = -(1-t^2)\gamma_n^2/4$ and thus

$$\left| \int_{\lambda_n^-}^{\tau_n} \frac{\lambda_n^\bullet - \lambda}{w_n(\lambda)} \chi_n(\lambda)\, \mathrm{d}\lambda \right| \leqslant \int_0^1 \frac{|\lambda_n^\bullet - \tau_n + t\gamma_n/2|}{\sqrt[+]{1-t^2}} |\chi_n(\lambda_t)|\, \mathrm{d}t \leqslant C|\gamma_n|,$$

where $C > 0$ can be chosen independently of k. Since $\gamma_n(\varphi_k) \to 0$ as $k \to \infty$, we conclude

$$F_n(\nu, \varphi_k) = \frac{1}{2\mathrm{i}} \int_{\tau_n}^{\nu} \frac{\lambda_n^\bullet - \lambda}{w_n(\lambda)} \chi_n(\lambda)\, \mathrm{d}\lambda + o(1), \qquad k \to \infty. \tag{F.3}$$

We can choose $\nu_* \in \partial U_n$ so that the straight line segment $[\tau_n, \nu_*]$ is admissible for any φ_k, $k \geqslant 1$. Note that $z_k \coloneqq \nu_* - \tau_n(\varphi_k) \neq 0$ for any $k \geqslant 1$ and $z_k \to z_\infty \coloneqq \nu_* - \tau_n(\varphi_\infty)$. With the parametrization $\lambda_t = \tau_n + t z_k$, $0 \leqslant t \leqslant 1$, and since by the definition of the standard root $w_n(\lambda_t) = -z_k \sqrt[+]{t^2 - \gamma_n^2/4z_k^2}$, one then obtains

$$\int_{\tau_n}^{\nu_*} \frac{\lambda_n^\bullet - \lambda}{w_n(\lambda)} \chi_n(\lambda)\, \mathrm{d}\lambda = z_k \int_0^1 \frac{t + (\tau_n - \lambda_n^\bullet)/z_k}{\sqrt[+]{t^2 - \gamma_n^2/4z_k^2}} \chi_n(\lambda_t)\, \mathrm{d}t.$$

As $\gamma_n(\varphi_k) \to 0$, one has $\sqrt[+]{t^2 - \gamma_n^2/4z_k^2}\big|_{\varphi_k} \to t$ and $t + (\tau_n - \lambda_n^\bullet)\big|_{\varphi_k} \to t$ by (F.2). To see that the integral converges to $\int_{\tau_n}^{\nu_*} \chi_n(\lambda)\, \mathrm{d}\lambda\big|_{\varphi_\infty}$ let $\varepsilon_k = \gamma_n(\varphi_k)/2z_k$. Then

$$\left| \sqrt[+]{t^2 - \varepsilon_k^2} \right| \geqslant \sqrt{t^2 - |\varepsilon_k|^2} \geqslant (t - |\varepsilon_k|)^{1/2}(t + |\varepsilon_k|)^{1/2},$$

whereas by (F.2) $|(\tau_n - \lambda_n^\bullet)/z_k| \leqslant |\varepsilon_k|$. Hence for any $0 \leqslant t \leqslant 1$

$$\left| \frac{t + (\tau_n - \lambda_n^\bullet)/z_k}{\sqrt[+]{t^2 - \gamma_n^2/4z_k^2}} \chi_n(\lambda_t) \right| \leqslant C g_k(t), \qquad g_k(t) = \frac{(t + |\varepsilon_k|)^{1/2}}{(t - |\varepsilon_k|)^{1/2}}.$$

Since g_k is in $L^1[0,1]$ for any $k \geqslant 1$ and converges in $L^1[0,1]$ to the constant function 1 as $k \to \infty$, it follows from the generalized dominated convergence theorem that

$$\begin{aligned} \lim_{k\to\infty} z_k \int_0^1 \frac{t + (\tau_n - \lambda_n^\bullet)/z_k}{\sqrt[+]{t^2 - \gamma_n^2/4z_k^2}} \chi_n(\lambda_t)\, \mathrm{d}t \bigg|_{\varphi_k} &= z_\infty \int_0^1 \chi_n(\lambda_t)\, \mathrm{d}t \bigg|_{\varphi_\infty} \\ &= \int_{\tau_n}^{\nu_*} \chi_n(\lambda)\, \mathrm{d}\lambda \bigg|_{\varphi_\infty}. \end{aligned}$$

In view of (F.3) it then follows that $\lim_{k\to\infty} F_n(\varphi_k) = F_n(\varphi_\infty)$. Altogether we have shown that F_n is continuous on V_φ. The claimed analyticity then follows from [23, Theorem A.6]. ▮

Corollary F.3 *Suppose $\varphi \in \mathcal{W}^p$, $1 < p < \infty$, and let $n, m \in \mathbb{Z}$. For any $\lambda_n \in \{\lambda_n^-, \lambda_n^+\}$ and $\lambda_m \in \{\lambda_m^-, \lambda_m^+\}$ the functional*

$$\psi \mapsto \int_{\lambda_n}^{\lambda_m} \omega$$

is analytic on V_φ. ⋊

G. A priori estimates of the Fundamental Solution

In this appendix we obtain, for smooth potentials, estimates of the gradients of several spectral quantities in the Fourier Lebesgue spaces. Such estimates are needed in the proof of Lemma 16.1 to show that the differential of the Birkhoff map is a compact perturbation of the identity. We derive these estimates from estimates of the gradient of the fundamental solution of the Zakharov-Shabat operator. According to [23, Theorem 1.1] the fundamental solution $M(t,\lambda,\varphi)$ of $L(\varphi)$ is continuous on $[0,\infty)\times\mathbb{C}\times L_c^2$ and for each fixed $t\in[0,\infty)$ is analytic in λ and φ. Further, we recall from [23, Section 1] the notation $L(\varphi) = R\partial_x + \Phi$ where

$$R = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix},$$

as well as

$$\hat{M}(t,\lambda,\varphi) \coloneqq M(t,\lambda,\varphi) - E_\lambda(t), \qquad E_\lambda(t) \coloneqq \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\lambda t} & \\ & \mathrm{e}^{\mathrm{i}\lambda t} \end{pmatrix}.$$

We also use the following exponentially weighted norms for t-dependent matrices A

$$\|A(t)\|_\lambda \coloneqq \mathrm{e}^{-|\Im\lambda|t}\|A(t)\|.$$

The following estimate is straightforward and can be found in [23, Lemma 2.1].

Lemma G.1 *On* $[0,1]\times\mathbb{C}\times L_c^2$

$$\|\hat{M}(t,\lambda)\|_\lambda \leqslant \|F(t,\lambda)\|_\lambda + c_\varphi\left(\int_0^t \|F(s,\lambda)\|_\lambda^2 \,\mathrm{d}s\right)^{1/2}.$$

with $F(t,\lambda) = \int_0^t E_\lambda(t-2s)\tilde{\Phi}(s)\,\mathrm{d}s$ *and* $c_\varphi = \|\varphi\|\mathrm{e}^{\|\varphi\|}$. ⋊

For $\varphi\in H_c^1$ one obtains the following estimate of F and hence $\hat{M}$ – see also [23, Theorem 2.3].

Lemma G.2 *For any* $\lambda\neq 0$ *and* $t\geqslant 0$

$$\|F(t,\lambda)\|_\lambda \leqslant \frac{2+\sqrt{t}}{2|\lambda|}\|\varphi\|_{H^1[0,t]}.$$

Consequently, on $[0,1]\times\mathbb{C}\setminus\{0\}\times H_c^1$

$$\|\hat{M}(t,\lambda)\|_\lambda \leqslant \frac{3}{2|\lambda|}(1+c_\varphi)\|\varphi\|_{H^1[0,1]}.$$ ⋊

Proof. Since $E_\lambda(t-2s) = \frac{1}{2\lambda}R\partial_s(E_\lambda(t-2s))$ for any $\lambda\neq 0$, we conclude with $\tilde{\Phi} = R\Phi$

$$F(t,\lambda) = \frac{1}{2\lambda}R\left(E_\lambda(-t)\tilde{\Phi}(t) - E_\lambda(t)\tilde{\Phi}(0) - \int_0^t E_\lambda(t-2s)\partial_s\tilde{\Phi}(s)\,\mathrm{d}s\right),$$

hence $\|F(t,\lambda)\|_\lambda \leqslant \frac{2+\sqrt{t}}{2|\lambda|}\|\varphi\|_{H^1[0,t]}$. ∎

We are now in a position to prove the following estimates for the fundamental solution in the Fourier Lebesgue spaces.

Lemma G.3 *Suppose that $\nu_n = n\pi + O(1)$, then*

$$\|\hat{M}(\nu_n)\|_{H^1[0,1]} = O(1),$$

uniformly in φ on bounded subsets of H^1_c and uniformly in n. Moreover for any $\varepsilon > 0$

$$\|\hat{M}(\nu_n)\|_{FL^q} = O(1/n^{\frac{q-1-\varepsilon}{1-\varepsilon}}), \qquad \forall\ 1+\varepsilon \leqslant q \leqslant 2.$$

If, in addition $\nu_n = n\pi + O(1/n)$, then

$$\|M(\nu_n) - E_{n\pi}\|_{FL^q} = O(1/n^{\frac{q-1-\varepsilon}{1-\varepsilon}}), \qquad \forall\ 1+\varepsilon \leqslant q \leqslant 2. \quad \rtimes$$

Proof. By a straightforward computation one obtains

$$\hat{M}(t,\lambda) = F(t,\lambda) + \int_0^t E_\lambda(t-s)\tilde{\Phi}(s)\hat{M}(s,\lambda)\ \mathrm{d}s.$$

Differentiating in t and using $\partial_t E_\lambda(t-s) = \lambda R E_\lambda(t-s)$ yields

$$\partial_t \hat{M}(t,\lambda) = \partial_t F(t,\lambda) + \tilde{\Phi}(t)\hat{M}(t,\lambda) + \lambda \int_0^t R E_\lambda(t-s)\tilde{\Phi}(s)\hat{M}(s,\lambda)\ \mathrm{d}s.$$

Consequently,

$$\begin{aligned}\|\partial_t \hat{M}(t,\lambda)\|_\lambda \leqslant \|\partial_t F(t,\lambda)\|_\lambda + \|\varphi\|_{H^1[0,t]}\|\hat{M}(t,\lambda)\|_\lambda \\ + |\lambda| \int_0^t |\varphi(s)|\|\hat{M}(s,\lambda)\|_\lambda\ \mathrm{d}s,\end{aligned}$$

and by Cauchy-Schwarz

$$\int_0^t |\varphi(s)|\|\hat{M}(s,\lambda)\|_\lambda\ \mathrm{d}s \leqslant \|\varphi\|_{L^2[0,t]}\left(\int_0^t \|\hat{M}(\lambda,s)\|_\lambda^2\ \mathrm{d}s\right)^{1/2}.$$

Moreover,

$$\begin{aligned}\partial_t F(t) &= E_\lambda(-t)\tilde{\Phi}(t) + \int_0^t \partial_t E_\lambda(t-2s)\tilde{\Phi}(s)\ \mathrm{d}s \\ &= E_\lambda(-t)\tilde{\Phi}(t) - \frac{1}{2}\int_0^t \partial_s E_\lambda(t-2s)\tilde{\Phi}(s)\ \mathrm{d}s \\ &= \frac{1}{2}E_\lambda(-t)\tilde{\Phi}(t) + \frac{1}{2}E_\lambda(t)\tilde{\Phi}(0) + \frac{1}{2}\int_0^t E_\lambda(t-2s)\partial_s\tilde{\Phi}(s)\ \mathrm{d}s,\end{aligned}$$

hence $\|\partial_t F(t)\|_\lambda \leqslant (1+\sqrt{t})\|\varphi\|_{H^1[0,t]}$. Altogether this shows that

$$\|\partial_t M(t,\lambda)\|_\lambda = O(1)$$

uniformly in $|\lambda| \geqslant 1$, $0 \leqslant t \leqslant 1$, and uniformly in φ on bounded subsets of H^1_c.

Suppose $\nu_n = n\pi + O(1)$, then $\sup_{0\leqslant t\leqslant 1}\|\hat{M}(\nu_n,t)\| = O(1/n)$ uniformly in φ on bounded subsets of H^1 and uniformly in n. Therefore,

$$\|\hat{M}(\nu_n)\|_{L^2[0,1]} = O(1/n), \qquad \|\hat{M}(\nu_n)\|_{H^1[0,1]} = O(1).$$

For any $\varepsilon > 0$ we have $H^1[0,1] \hookrightarrow FL^{1+\varepsilon}$ by Lemma A.9, hence it follows by interpolation that

$$\|\hat{M}(\nu_n)\|_{FL^q} = O\left(\frac{1}{n^{\frac{q-1-\varepsilon}{1-\varepsilon}}}\right), \qquad \forall\ 1+\varepsilon \leqslant q \leqslant 2,$$

uniformly in φ on bounded subsets of H^1 and uniformly in n.

Notice that if $|\nu_n - n\pi| \leqslant \pi/4$ for n sufficiently large, then $\|E_{\nu_n} - E_{n\pi}\|_{FL^q} = O(1)$ for any $q > 1$ uniformly in n by Lemma E.3. Moreover, $\|E_{\nu_n} - E_{n\pi}\|_{L^2} = O(|\nu_n - n\pi|)$. Thus, the final claim follows again by interpolation. ∎

Corollary G.4 *Suppose that $|\nu_n - n\pi| \leqslant \pi/4$ for $|n|$ sufficiently large and $1 < p < \infty$ then*

$$\|\hat{M}(\nu_n)\|_{FL^q} = \ell^p_n,$$

uniformly on bounded subsets of H^1_c for each fixed $q > 1 + \frac{1}{p}$. Moreover, if $\nu_n = n\pi + O(1/n)$, then also

$$\|M(\nu_n) - E_{n\pi}\|_{FL^q} = \ell^p_n. \quad \rtimes$$

Proof. By the preceding Lemma $\|\hat{M}(\nu_n)\|_{FL^q} = O(1/n^{\frac{q-1-\varepsilon}{1-\varepsilon}})$ for any $\varepsilon > 0$ with $1 + \varepsilon \leqslant q \leqslant 2$. Moreover, $O(1/n^{\frac{q-1-\varepsilon}{1-\varepsilon}}) = \ell^p_n$ is tantamount to

$$q > 1 + \frac{1}{p} + \varepsilon\left(1 - \frac{1}{p}\right).$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, $\|\hat{M}(\nu_n)\|_{FL^q} = \ell^p_n$ for $q > 1 + 1/p$. The second claim follows in similar fashion. ∎

Our next result gives an estimate of the gradient of the fundamental solution. To state it, we recall from Section 2 the definition[2]

$$e^-_\alpha = (\mathrm{e}_{-\alpha}, 0), \quad e^+_\alpha = (0, \mathrm{e}_\alpha), \qquad \mathrm{e}_\alpha(x) = \mathrm{e}^{\mathrm{i}\alpha\pi x}.$$

Lemma G.5 *Suppose $\varphi \in H^1_c$. Then for any sequence with $|\nu_n - n\pi| \leqslant \pi/4$ and any $2 \leqslant p \leqslant \infty$*

$$\left\| \mathrm{i}\partial\grave{M}\,|_{\lambda=\nu_n} - \grave{E}_{\nu_n}\begin{pmatrix} 0 & -e^+_{-2\nu_n/\pi} \\ e^-_{-2\nu_n/\pi} & 0 \end{pmatrix} \right\|_{FL^{p'}} = \ell^p_n.$$

If, in addition, $\nu_n = n\pi + O(1/n)$, then also

$$\left\| \mathrm{i}\partial\grave{M}\,|_{\lambda=\nu_n} - (-1)^n\begin{pmatrix} 0 & -e^+_{-2n} \\ e^-_{-2n} & 0 \end{pmatrix} \right\|_{FL^{p'}} = \ell^p_n. \quad \rtimes$$

Proof. The gradient is given by

$$\mathrm{i}\partial\grave{M} = \grave{M}\begin{pmatrix} -M_1 \star M_2 & -M_2 \star M_2 \\ M_1 \star M_1 & M_1 \star M_2 \end{pmatrix},$$

where $\grave{M} = \grave{E}_{\nu_n} + \ell^2_n$. We denote

$$\hat{M}_1 = M_1 - e^-_{\nu_n/\pi}, \quad \hat{M}_2 = M_2 - e^+_{\nu_n/\pi},$$

and furthermore write

$$M_1 \star M_1 = e^+_{-2\nu_n/\pi} + 2\hat{M}_1 \star e^-_{\nu_n/\pi} + \hat{M}_1 \star \hat{M}_1,$$

$$M_1 \star M_2 = \hat{M}_2 \star e^-_{\nu_n/\pi} + \hat{M}_1 \star e^+_{\nu_n/\pi} + \hat{M}_1 \star \hat{M}_2,$$

$$M_2 \star M_2 = e^-_{-2\nu_n/\pi} + 2\hat{M}_2 \star e^+_{\nu_n/\pi} + \hat{M}_2 \star \hat{M}_2.$$

By Young's inequality for $a, b \geqslant 1$ with $1 + 1/p' = 1/a + 1/b$

$$\|\hat{M}_1 \star e^\pm_{\nu_n/\pi}\|_{FL^{p'}} \leqslant \|\hat{M}_1\|_{FL^a}\|e^\pm_{\nu_n/\pi}\|_{FL^b}.$$

We may choose $b > 1$ and $a = p' - \delta$ with some $\delta > 0$ arbitrary small. Note that $\|e^\pm_{\nu_n/\pi}\|_{FL^b} = O(1)$ for any $b > 1$ by Lemma E.3. Moreover, $p' = 1 + 1/(p-1) > 1 + 1/p$, hence also $p' - \delta > 1 + 1/p$ if δ is chosen sufficiently small. Thus $\|\hat{M}_1\|_{FL^a} = \ell^p_n$ by Corollary G.4. One argues analogously in the remaining cases which proves the first claim.

If $\nu_n = n\pi + O(1/n)$, then $\grave{E}_{\nu_n} = (-1)^n I + \ell^{1+}_n$ and $\|e^\pm_{\mp 2\nu_n/\pi} - e_{\mp 2n}\|_{FL^{p'}} = \ell^p_n$ which proves the second claim. ∎

[2] We point out that our notation of $e^\pm_n$ is different from the one in [23].

As an immediate corollary we obtain the following estimates for the gradient of the discriminant Δ, which is the trace of the fundamental solution, and the gradient of the anti-discriminant δ, which is the anti-trace of the fundamental solution – see [23, Lemma 4.4] for the case $p = 2$.

Corollary G.6 *Suppose $\varphi \in H_c^1$. Then for any sequence with $|\nu_n - n\pi| \leqslant \pi/4$ and any $2 \leqslant p < \infty$*

$$\|\mathrm{i}\partial\Delta|_{\lambda=\nu_n}\|_{FL^{p'}} = \ell_n^p.$$

If, in addition $\nu_n = n\pi + O(1/n)$ then

$$\|\mathrm{i}\partial\delta|_{\lambda=\nu_n} - (-1)^n(e_{-2n\pi}^+ - e_{-2n\pi}^-)\|_{FL^{p'}} = \ell_n^p.$$

The mid-points τ_n, $n \in \mathbb{Z}$, and the Dirichlet eigenvalues μ_n, $n \in \mathbb{Z}$, are real analytic on L_r^2 by Lemma 10.2. We adapt [23, Lemma 12.5] to obtain an estimate of their gradients for the case $p \neq 2$.

Lemma G.7 *At any potential $\varphi \in \mathcal{W} \cap H_c^1$*

$$\|\partial\tau_n\|_{FL^{p'}} = \ell_n^p, \qquad \|\partial\mu_n - \frac{1}{2}(e_{-2n\pi}^+ + e_{-2n\pi}^-)\|_{FL^{p'}} = \ell_n^p. \qquad \rtimes$$

Proof. We recall from [23, Lemma 12.4] that

$$\partial\tau_n = -\frac{1}{2\pi\mathrm{i}}\int_{\partial U_n} \frac{\Delta(\lambda)\partial\Delta(\lambda)}{\Delta^2(\lambda) - 4}\,\mathrm{d}\lambda,$$

where $U_n = D_n = \{\lambda \in \mathbb{C} : |\lambda - n\pi| < \pi/4\}$ for $|n|$ sufficiently large. Since $\sup_{\lambda\in\partial U_n}\left|\frac{\Delta(\lambda)}{\Delta^2(\lambda)-4}\right| = O(1)$ and $\sup_{\lambda\in\partial U_n}\|\partial\Delta(\lambda)\|_{FL^{p'}} = \ell_n^p$ the claim for $\partial\tau_n$ follows.

On $\mathcal{W}$ the Dirichlet eigenvalues are simple and hence locally analytic. To obtain the estimate of their gradient, we follow [23, Section 7], let $g = M_1 + M_2$, and choose for any $n \in \mathbb{Z}$ the canonical Dirichlet eigenfunction

$$g_n \coloneqq g|_{\lambda=\mu_n}.$$

Then in view of [23, Proposition 7.5] the gradient has the form

$$\partial\mu_n = \frac{g_n \star g_n}{Q(g_n)}, \qquad Q(f) = 2\int_{\mathbb{T}} g_- g_+\,\mathrm{d}x.$$

On $\mathcal{W} \cap H_c^1$ the claimed estimates now follow from Lemma G.5. ∎

H. Properties of the NLS hierarchy

In this appendix we recall some well known facts about the Hamiltonians of the NLS hierarchy – see [23, Section 4] as well as [39]. For $\varphi = (\varphi_-, \varphi_+) \in H_c^{k-1}$ the kth NLS Hamiltonian is given by,

$$\mathcal{H}_k(\varphi) = (-\mathrm{i})^{k+1}\int_{\mathbb{T}} \varphi_- u_k(\varphi_-, \varphi_+, \dots, \varphi_-^{(k-1)}, \varphi_+^{(k-1)})\,\mathrm{d}x, \qquad k \geqslant 1,$$

where $u_1 = -\varphi_+$, and

$$u_{k+1} = u_k' + \varphi_- \textstyle\sum_{l=1}^{k-1} u_{k-l}u_l, \qquad k \geqslant 1.$$

Lemma H.1 *If $\varphi \in H_c^{k-1}$, then*

$$u_{k+1} = -\varphi_+^{(k)} + q_k(\varphi_-, \varphi_+, \dots, \varphi_-^{(k-2)}, \varphi_+^{(k-2)}),$$

where q_k is a homogeneous polynomial of degree $k+1$ when φ_-, φ_+, and ∂_x each count as one degree. Moreover, each term of $\varphi_- q_k$ has at most degree $k-2$ with respect to ∂_x, and the degree with respect to φ_- equals the one with respect to φ_+. ⋊

Proof. As is evident from their definition, the polynomials u_k are homogeneous of degree k, and only contain derivatives of φ_- and φ_+ up to order $k-1$. Furthermore, $u_k(\varphi_-, \lambda\varphi_+) = \lambda u_k(\lambda\varphi_-, \varphi_+)$ for all $\lambda \in \mathbb{C}$, which completes the proof. ∎

In case of a smooth real type potential, that is $\varphi = (\psi, \overline{\psi})$, the odd Hamiltonians have the form

$$\mathcal{H}_{2m+1}(\varphi) = \int_{\mathbb{T}} \left(|\psi^{(m)}|^2 + \psi q_{2m} \right) \mathrm{d}x, \qquad m \geqslant 1,$$

where q_{2m} depends on ψ, $\overline{\psi}$, and their derivatives up to order $2m-2$. Suppose ψq_{2m} contains a monomial $\psi_{(m+n)}\mathfrak{q}(\psi, \overline{\psi}, \dots, \psi_{(2m-2)}, \overline{\psi}_{(2m-2)})$ with $n \geqslant 0$. Since this monomial has at most degree $2m-2$ with respect to ∂_x, it follows that $\mathfrak{q}$ contains at most $m-2-n$ derivatives. Thus we can integrate by parts to the effect that each factor of the monomial contains at most $m-1$ derivatives.

Corollary H.2 *For any $m \geqslant 1$ there exists a polynomial p_{2m} such that*

$$\mathcal{H}_{2m+1}(\varphi) = \int_{\mathbb{T}} \left(|\psi_{(m)}|^2 + p_{2m}(\psi, \overline{\psi}, \dots, \psi_{(m-1)}, \overline{\psi}_{(m-1)}) \right) \mathrm{d}x,$$

for all $\varphi = (\psi, \overline{\psi})$ in H_r^m. The polynomial p_{2m} is homogenous of degree $2m+2$ with respect to ψ, $\overline{\psi}$, and ∂_x, and the degree of each term of p_{2m} with respect to ψ equals the one with respect to $\overline{\psi}$. ⋊

I. A diffeomorphism property

Let Z be a Banach space and $T\colon Z \to Z$ be a bounded operator. Suppose $Z = X \oplus Y$ and suppose T admits with respect to this direct sum the decomposition

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{I.1}$$

Lemma I.1 (Schur complement) *The operator $\mathrm{Id}_Z + T$ is invertible on Z if and only if $\mathrm{Id}_Y + D$ is invertible on Y and the Schur complement*

$$S = \mathrm{Id}_X + A - B(\mathrm{Id}_Y + D)^{-1}C$$

is invertible on X. ⋊

As an immediate corollary we obtain the following sufficient condition.

Corollary I.2 *Suppose X is finite dimensional, then $\mathrm{Id}_Z + T$ is invertible if*

$$\det S \neq 0, \qquad \|D\|_{L(Y)} < 1. \quad ⋊$$

A subset K of Z is compact if and only if it is totally bounded and complete. The following characterization of totally bounded sets in ℓ^p is well known.

Lemma I.3 *A subset B of ℓ^p, $1 \leqslant p < \infty$, is totally bounded if and only if it is pointwise bounded and for any $\varepsilon > 0$ there exists an $N \geqslant 1$ so that $\|\pi_N^\perp x\|_p \leqslant \varepsilon$ for all $x \in B$.* ⋊

We now apply this characterization of totally bounded sets and the Schur complement to prove the following diffeomorphism property.

Proposition I.4 *Suppose $\mathcal{V}^{s,p}$ is an open, path connected subset of $\ell_+^{s,p}$ and $f\colon \mathcal{V}^{s,p} \to \ell^{s,p}$, $1 \leqslant p < \infty$, $s \in \mathbb{R}$, is a real analytic map with the properties that*

(i) $\mathrm{d}_x f - \mathrm{Id}\colon \ell_{\mathbb{C}}^{s,p} \to \ell_{\mathbb{C}}^{s,p}$ *is compact for any $x \in \mathcal{V}^{s,p}$,*

(ii) *f is a local diffeomorphism at some point of $\mathcal{V}^{s,p}$.*

Then f is a local diffeomorphism generically on $\mathcal{V}^{s,p}$. ⋊

Proof. To simplify notation write $T_z = \mathrm{d}_z f - \mathrm{Id}_{\ell_{\mathbb{C}}^{s,p}}$. By assumption (i) T_z is a compact operator on $\ell_{\mathbb{C}}^{s,p}$ for any $z \in \mathcal{V}^{s,p}$. In particular, the image of the unitball in $\ell_{\mathbb{C}}^{s,p}$ is relatively compact in $\ell_{\mathbb{C}}^{s,p}$. By Lemma I.3 there exists $N \geqslant 1$ (which depends on z) so that $\|\pi_N^\perp T_z\|_{L(\ell_{\mathbb{C}}^{s,p})} \leqslant 1/4$. Since $\|\pi_N^\perp T_z\|_{L(\ell_{\mathbb{C}}^{s,p})}$ depends continuously on z, there exists an complex neighborhood V of z within $\ell_{\mathbb{C}}^{s,p}$ so that $\|\pi_N^\perp T_w\|_{L(\ell_{\mathbb{C}}^{s,p})} \leqslant 1/2$ for all $w \in V$.

Let W be any nontrivial open subset of $\mathcal{V}^{s,p}$ and denote by $z_0 \in \mathcal{V}^{s,p}$ the point of assumption (ii) at which the differential of f is invertible. For any $z_1 \in W$ the path Γ connecting z_0 and z_1 is compact in $\mathcal{V}^{s,p}$ and hence can be covered by finitely many neighborhoods V as constructed above. Consequently, there exists a complex neighborhood U of Γ within $\mathcal{V}^{s,p}$ and an integer $N_U \geqslant 1$ so that

$$\|\pi_{N_U}^\perp T_z\|_{L(\ell_{\mathbb{C}}^{s,p})} \leqslant 1/2, \qquad \forall\ z \in U.$$

Decompose $\ell_{\mathbb{C}}^{s,p} = X^{N_U} \oplus Y^{N_U}$ where $X^{N_U} = \pi_{N_U}(\ell_{\mathbb{C}}^{s,p})$ and $Y^{N_U} = \pi_{N_U}^\perp(\ell_{\mathbb{C}}^{s,p})$. We can decompose for any $z \in U$ the operator T_z according to (I.1). Since for any $z \in U$

$$\|D_z^{N_U}\|_{L(X^{N_U})} \leqslant \|\pi_{N_U}^\perp T_z\|_{L(\ell_{\mathbb{C}}^{s,p})} \leqslant 1/2,$$

by Corollary I.2 the differential $\mathrm{d}_z f$ is invertible for all $z \in U$ with

$$\lambda(z) = \det S_z^{N_U} \neq 0.$$

Note that

$$U \to L(X^{N_U}), \qquad z \mapsto S_z^{N_U} = \mathrm{Id}_{X^{N_U}} - A_z^{N_U} + B_z^{N_U}(\mathrm{Id}_{Y^{N_U}} + D_z^{N_U})^{-1} C_z^{N_U},$$

is analytic, hence the function $\lambda\colon U \to \mathbb{C}$ is analytic. Since $\lambda(z_0) \neq 0$ it follows from the identity theorem that λ does not vanish on any open subset of X. In particular, λ does not vanish identically on W. Consequently, the set $\Lambda = \{z \in \mathcal{V}^{s,p} : \mathrm{d}_z f \text{ is invertible}\}$ has nontrivial intersection with W. Since W was arbitrary, it follows that Λ is dense in $\mathcal{V}^{s,p}$. Since Λ is open the claim follows. ∎

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Notations

Spaces.

$$FL_c^{s,p}(\mathbb{T}_a) = FL^{s,p}(\mathbb{T}_a,\mathbb{C})\times FL^{s,p}(\mathbb{T}_a,\mathbb{C})$$
$$FL_r^{s,p}(\mathbb{T}_a) = \{\varphi\in FL_c^{s,p}(\mathbb{T}_a) : \varphi_+ = \overline{\varphi_-}\}$$
$$FL_c^p = FL_c^{0,p}(\mathbb{T}_1),\quad FL_r^p = FL_r^{0,p}(\mathbb{T}_1)$$
$$H_c^s = FL_c^{s,2}(\mathbb{T}_1),\quad H_r^s = FL_r^{s,2}(\mathbb{T}_1)$$
$$\ell_c^{s,p} = \ell^{s,p}(\mathbb{Z},\mathbb{C})\times\ell^{s,p}(\mathbb{Z},\mathbb{C})$$
$$\ell_r^{s,p} = \ell^{s,p}(\mathbb{Z},\mathbb{R})\times\ell^{s,p}(\mathbb{Z},\mathbb{R})$$
$$\ell_+^{s,p} = \{I\in\ell^{s,p}(\mathbb{Z},\mathbb{R}) : I_m \geqslant 0\ \forall\ m\in\mathbb{Z}\}$$

$\mathcal{W}^p$ complex neighborhood of FL_r^p

$\hat{\mathcal{W}}^p$ defined in Lemma 10.1

$$\mathfrak{U}_n = \{\lambda\in\mathbb{C} : |\Re\lambda - n\pi| \leqslant \pi/2\}$$

U_n = isolating disc around G_n

$$D_n = \{\lambda\in\mathbb{C} : |\lambda - n\pi| < \pi/4\}$$
$$Z_n = \{\varphi\in\hat{\mathcal{W}}^p : \gamma_n(\varphi) = 0\}$$
$$\operatorname{Iso}(\varphi) = \{\psi\in FL_r^p : \Delta(\cdot,\psi) = \Delta(\cdot,\varphi)\}$$

Spectral quantities.

$\lambda_n^\pm$ periodic eigenvalues

$$G_n = [\lambda_n^-,\lambda_n^+]$$

μ_n = Dirichlet eigenvalues

$$\gamma_n = \lambda_n^+ - \lambda_n^-$$
$$\tau_n = (\lambda_n^+ + \lambda_n^-)/2$$

Δ discriminant – Section 8

δ anti-discriminant – Section 9

Miscellaneous.

$$\mathcal{H}_1 = \int_{\mathbb{T}} \varphi_-\varphi_+\,\mathrm{d}x,$$
$$\mathcal{H} = \mathcal{H}_3 = \int_{\mathbb{T}} (\varphi_-'\varphi_+' + \varphi_-^2\varphi_+^2)\,\mathrm{d}x,$$
$$\xi_n = \sqrt[+]{4I_n/\gamma_n^2}$$
$$z_n^\pm = \gamma_n \mathrm{e}^{\pm\mathrm{i}\eta_n}$$
$$\sigma^0 = (n\pi)_{n\in\mathbb{Z}}$$
$$\pi_n = \begin{cases} n\pi, & n\neq 0,\\ 1, & n = 0.\end{cases}$$
$$F_n(\lambda) = \int_{\lambda_n^+}^{\lambda} \frac{\Delta^\bullet(\mu)}{\sqrt[c]{\Delta^2(\mu)-4}}\,\mathrm{d}\mu$$
$$F(\lambda) = F_0(\lambda)$$

Operators.

$$L(\varphi) = \begin{pmatrix}\mathrm{i} & \\ & -\mathrm{i}\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_-\\ \varphi_+ & \end{pmatrix}$$
$$\Phi = \begin{pmatrix} & \varphi_-\\ \varphi_+ & \end{pmatrix}$$
$$P = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$$
$$R = \begin{pmatrix}\mathrm{i} & \\ & -\mathrm{i}\end{pmatrix}$$
$$J = \begin{pmatrix} & 1\\ -1 & \end{pmatrix}$$
$$T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}$$
$$e_n^+ = \begin{pmatrix}0\\ \mathrm{e}^{\mathrm{i}n\pi x}\end{pmatrix}$$
$$e_n^- = \begin{pmatrix}\mathrm{e}^{-\mathrm{i}n\pi x}\\ 0\end{pmatrix}$$

Operations.

$$\dot{f} = \partial_\lambda f$$

$\mathrm{d}f$ differential of f

∂f gradient of f

$\langle\varphi,\psi\rangle$ sesquilinear dual product – Section 2

$\langle\varphi,\psi\rangle_r$ real dual product – Section 2

$$\{F,G\} = -\mathrm{i}\langle\partial F, J\partial G\rangle_r$$

Products.

$$\Delta^2(\lambda) - 4 = -4\prod_{m\in\mathbb{Z}}\frac{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}{\pi_m^2}$$
$$\Delta^\bullet(\lambda) = 2\prod_{m\in\mathbb{Z}}\frac{\lambda_m^\bullet - \lambda}{\pi_m}$$
$$\psi_n(\lambda) = -\frac{2}{\pi_n}\prod_{m\neq n}\frac{\sigma_m^n - \lambda}{\pi_m}$$
$$\zeta_m(\lambda) = \prod_{k\neq m}\frac{\sigma_k^n - \lambda}{w_k(\lambda)}$$
$$\zeta_m^r(\lambda) = \frac{\mathrm{i}}{w_r(\lambda)}\prod_{k\neq m,r}\frac{\sigma_k^n - \lambda}{w_k(\lambda)}$$
$$\chi_n(\lambda) = \prod_{m\neq n}\frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}$$
$$\sin\lambda = -\prod_{m\in\mathbb{Z}}\frac{m\pi - \lambda}{\pi_m}$$

Roots.

$$w_n(\lambda) = (\tau_n - \lambda)\sqrt[+]{1 - \gamma_n^2/4(\tau_n - \lambda)^2}$$
$$\sqrt[c]{\Delta^2(\lambda) - 4} = 2\mathrm{i}\prod_{m\in\mathbb{Z}}\frac{w_m(\lambda)}{\pi_m}$$
$$\sqrt[*]{\Delta^2(\mu_n) - 4} = \delta(\mu_n)$$