Spectral triangles of Zakharov-Shabat operators in weighted Sobolev spaces

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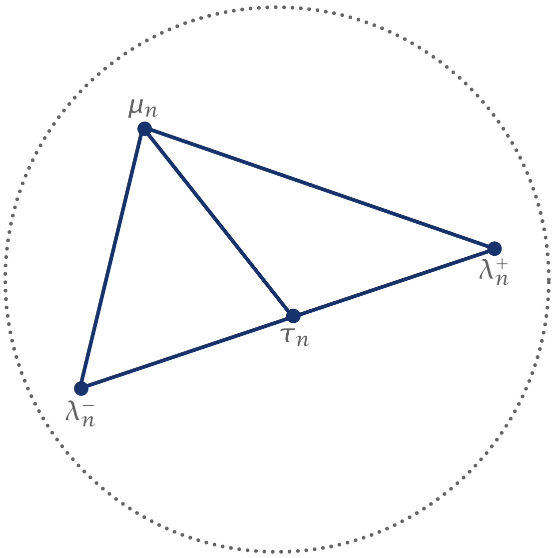
December 19, 2011



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1 Overview

The *nonlinear Schrödinger equation* on the real line

$$\mathrm{i}\partial_t u = -\partial_x^2 u + 2\alpha|u|^2 u, \qquad x \in \mathbb{R}, \quad \alpha = \pm 1,$$

is a cubic perturbation of the wave function of a free one-dimensional particle. The case $\alpha = 1$ is called *defocusing* and $\alpha = -1$ *focusing*. This evolution equation provides a suitable setting to describe slow modulations of a localized wave packet in dispersive media, and arises in many applications such as in hydrodynamics, nonlinear optics, nonlinear acoustics, plasma waves and biomolecular dynamics.

The linear dispersion, which tends to spread the wave packet, is balanced by the cubic non-linearity describing the self interaction of the wave with itself. This allows one of the most successful applications of the NLS equation, the description of soliton waves in optical fibers.

After the KdV equation, the NLS equation was the second evolution equation known to be integrable by the inverse scattering method [27]. When considered with periodic boundary conditions

$$u(x+1) = u(x),$$

it can be written in a hamiltonian form and admits a Lax-pair formalism. This enables us to view solutions of the NLS equation as the potentials of Zakharov-Shabat operators. The spectrum of the latter encodes many properties of the NLS solutions. Instead of analyzing the NLS equation directly, we thus focus on these much simpler linear operators.

In this treatise we pursue the assertion that the regularity of such a solution is closely related to the asymptotic behaviour of certain spectral data of the associated operator. More precisely, the Fourier coefficients of the solution and the associated spectral gap lengths belong to the same weighted Sobolev space, hence they share the same asymptotic behaviour. Since the spectrum of the associated Zakharov-Shabat operator is invariant under the NLS flow, a solution of the NLS equation stays in the same weighted Sobolev space for all time. One can use this relationship to show that the Birkhoff coordinates, which linearize the NLS flow [12, 22], preserve regularity. As integrating the NLS equation in these coordinates is trivial, well-posedness of the NLS equation follows in all weighted Sobolev spaces with subexponential weight. This encompasses high regularity spaces such as standard Sobolev spaces and Gevrey functions.

Hamiltonian formalism & Lax pair representation

To set the stage for the NLS hamiltonian system, let $L^2 \coloneqq L^2(\mathbb{T},\mathbb{C})$ denote the space of 1-periodic, L^2-integrable, complex functions on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. As phase space we introduce $\mathcal{L}^2 \coloneqq L^2 \times L^2$, with elements $\varphi = (\varphi_-, \varphi_+)$, and associated Poisson structure given by

$$\{F,G\} \coloneqq -\mathrm{i}\int_{\mathbb{T}} (\partial_{\varphi_-} F\,\partial_{\varphi_+} G - \partial_{\varphi_+} F\,\partial_{\varphi_-} G)\,\mathrm{d}x,$$

where $\partial_{\varphi_-} F$ and $\partial_{\varphi_+} F$ denote the components of the L^2-gradient $\partial_\varphi F$ of a C^1-functional F.

The hamiltonian system with Hamiltonian

$$H_{\text{NLS}} = \int_{\mathbb{T}} (\partial_x \varphi_- \partial_x \varphi_+ + \varphi_-^2 \varphi_+^2)\mathrm{d}x$$

is then given by

$$\begin{aligned} \mathrm{i}\partial_t \varphi_- &= \partial_{\varphi_+} H_{\text{NLS}} = -\partial_x^2 \varphi_- + 2\varphi_+ \varphi_-^2, \\ \mathrm{i}\partial_t \varphi_+ &= -\partial_{\varphi_-} H_{\text{NLS}} = \partial_x^2 \varphi_+ - 2\varphi_- \varphi_+^2, \end{aligned} \tag{$\star$}$$

which we also call the general NLS equation.

The defocusing NLS equation is obtained, when the general NLS equation is restricted to the invariant subspace of states of *real type*

$$\mathcal{L}_r^2 \coloneqq \left\{\varphi \in \mathcal{L}^2 : \varphi^* = \varphi\right\}, \qquad \varphi^* = (\overline{\varphi_+}, \overline{\varphi_-}).$$

On the other hand, we obtain the focusing NLS equation by restriction to the invariant subspace of states of *imaginary type*

$$\mathcal{L}_i^2 \coloneqq \left\{\varphi \in \mathcal{L}^2 : \varphi^* = -\varphi\right\}.$$

Note that the NLS Hamiltonian requires some regularity to be well defined, for example the standard Sobolev space $\mathcal{H}^m$ with $m \geqslant 1$. However, the initial value problem for the NLS equation on the circle is well posed on $\mathcal{L}^2$ as well – see [2].

Zakharov & Shabat [27] discovered a Lax-pair for the NLS equation. This enables us to view a solution of the general NLS equation as the potential of a linear operator. More precisely, we consider $\varphi = (\varphi_-, \varphi_+)$ as the potential of the Zakharov-Shabat operator

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix},$$

where blank matrix entries represent 0. A tedious computation yields that φ is a solution of the NLS equation ($\star$) if and only if

$$\partial_t L(\varphi) = [A(\varphi), L(\varphi)],$$

where the second operator of the Lax-pair is given by

$$A(\varphi) = \begin{pmatrix} 2\mathrm{i}\partial_x^2 - \mathrm{i}\varphi_+\varphi_- & \partial_x\varphi_- + 2\varphi_-\partial_x \\ \partial_x\varphi_+ + 2\varphi_+\partial_x & -2\mathrm{i}\partial_x^2 + \mathrm{i}\varphi_-\varphi_+ \end{pmatrix}.$$

Many properties of NLS solutions φ can be recovered from spectral data of $L(\varphi)$. Of particular interest is that the NLS flow leaves the periodic spectrum of this operator invariant, so many important features of the NLS dynamics are revealed by the static spectrum of $L(\varphi)$. We thus focus on these linear Zakharov-Shabat operators, instead of analyzing the NLS equation directly.

Spectrum

The periodic spectrum of $L(\varphi)$ is well known to be pure point and more precisely [22] to consist of a sequence of pairs of complex eigenvalues $\lambda_n^+(\varphi)$ and $\lambda_n^-(\varphi)$, listed with multiplicities, such that

$$\lambda_n^\pm(\varphi) = n\pi + \ell_n^2, \qquad n \in \mathbb{Z},$$

locally uniformly in φ. Here, ℓ_n^2 denotes a generic ℓ^2-sequence.

If we employ a lexicographical ordering on the complex numbers, ordering them first by their real and second by their imaginary part, the periodic spectrum is given by an unbounded bi-infinite sequence of eigenvalues, such that

$$\dots \preccurlyeq \lambda_{n-1}^+ \preccurlyeq \lambda_n^- \preccurlyeq \lambda_n^+ \preccurlyeq \lambda_{n+1}^- \preccurlyeq \dots,$$

where equality or inequality may occur in every place.

Suppose the potential φ is of real type as in the case of the defocusing NLS equation. Then $L(\varphi)$ is self-adjoint and so the periodic spectrum is real. By Floquet theory, the periodic eigenvalues characterise the spectrum of L when considered on the real line without boundary conditions. In this case the spectrum is absolutely continuous and given by

$$\operatorname{spec}_{\mathbb{R}}(L) = \mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} (\lambda_n^-, \lambda_n^+).$$

The complementary and possibly empty intervals $(\lambda_n^-, \lambda_n^+)$ precisely describe the gaps in the continuous spectrum, hence one speaks of *spectral gaps*. The length $\gamma_n \coloneqq \lambda_n^+ - \lambda_n^-$ is called the *nth spectral gap length* and we denote by $\gamma(\varphi) \coloneqq (\gamma_n)_{n\in\mathbb{Z}}$ the two sided sequence of spectral gap lengths of the potential φ. When some gap length vanishes, the corresponding gap is called *collapsed*, and otherwise *open*.

In contrast, for a general potential φ, the operator $L(\varphi)$ is not necessarily formally self-adjoint. The gap lengths are defined as before, however, they are no longer real, but complex numbers. They also cannot be interpreted as lengths of gaps in the continuous spectrum, when $L(\varphi)$ is considered on $\mathbb{R}$. Nevertheless we will call them gap lengths, and speak of open and collapsed gaps.

Statement of results

Our main concern is the relationship between the regularity of the potential φ and the asymptotic behaviour of γ_n. This question has a long history in the case of Hills' operator $\partial_x^2 + q$ with q being a 1-periodic real L^2-potential. Its spectrum is real, pure point, and consists of an unbounded sequence of periodic eigenvalues

$$\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \dots,$$

with gap lengths defined by $\gamma_n = \lambda_{2n} - \lambda_{2n-1}$. Marčenko & Ostrovskiĭ [18] observed that

$$q \in H^k(S^1, \mathbb{R}) \Leftrightarrow \sum_{n\geqslant 1} n^{2k}\gamma_n^2 < \infty,$$

for nonnegative integers k. Even earlier, Hochstadt [11] showed that

$$q \in C^\infty(S^1, \mathbb{R}) \Leftrightarrow \gamma_n = O(n^{-k}) \text{ for all } k \geqslant 0,$$

and Trubowitz [26] then proved that

$$q \in C^\omega(S^1, \mathbb{R}) \Leftrightarrow \gamma_n = O(\mathrm{e}^{-an}) \text{ for some } a > 0.$$

When the KdV flow was realized as an isospectral deformation of Hill's operator, other regularity classes, such as Gevrey-functions, have been considered, as well as complex potentials q. We point the reader to Pöschel [20, 21] for a self-contained presentation of recent progress.

Analogous results for the Zakharov-Shabat system are due to Grébert & Kappeler [9, 10], Djakov & Mityagin [5–7], and more recently Kappeler, Serier & Topalov [17]. The technical details are a bit more complicated than in Hill's case, mainly due to the fact that the distance of two distinct eigenvalues is bounded. The bottom line, however, is the same as for Hill's operator, the gap lengths characterise the regularity of the potential just like the Fourier coefficients do.

To allow a more precise statement, we introduce weighted Sobolev spaces [13, 14]. A *submultiplicative, and normalized weight* is a symmetric function $w\colon \mathbb{Z} \to \mathbb{R}$ with

$$w_n \geqslant 1, \qquad w_{n+m} \leqslant w_n w_m,$$

for all n and m. The class of all such weights is denoted by $\mathcal{M}$ and H^w is the Hilbert space of complex 1-periodic functions $u = \sum_{k\in\mathbb{Z}} u_k \mathrm{e}^{2\pi i k x}$ with finite w-norm

$$\|u\|_w^2 := \sum_{k\in\mathbb{Z}} w_{2k}^2 |u_k|^2.$$

We write $\mathcal{H}^w := H^w \times H^w$ for the 2-vector version, and denote by h^w the analog for complex sequences $u = (u_k)$, where

$$\|(u_k)_{k\in\mathbb{Z}}\|_w^2 := \sum_{k\in\mathbb{Z}} w_{2k}^2 |u_k|^2.$$

The forward problem, that is controlling the asymptotic behaviour of the gap lengths by the regularity of the potential, is analyzed in chapter 1. Using the language of weighted Sobolev spaces our result may be stated as follows.

Theorem 1 *If $\varphi \in \mathcal{H}^w$ and $w \in \mathcal{M}$, then $\gamma(\varphi) \in h^w$.* ⋊

Chapter 2 is then dedicated to the inverse problem of recovering the regularity of the potential from the asymptotic behaviour of its gap lengths. To obtain a true converse to Theorem 1, we first have to exclude weights of exponential growth, as there exist potentials with finitely many open gaps, so called *finite gap potentials*, that have poles and therefore are not entire functions. Second, we have to restrict ourselves to the subclass of potentials of real or imaginary type, to get a first inverse result.

Theorem 2 *Let $w \in \mathcal{M}$ and φ be of real or imaginary type with $\gamma(\varphi) \in h^w$.*

a) If w is subexponential, then $\varphi \in \mathcal{H}^w$.

b) If w is exponential, then $\varphi \in \mathcal{H}^{w_\varepsilon}$, for some $\varepsilon > 0$, where $w_\varepsilon = \mathrm{e}^{\varepsilon|\cdot|}$. ⋊

Theorem 1 is due to Grébert & Kappeler [9, 10] and can be found in the generality stated here in [6]. The inverse result was obtained by Djakov & Mityagin [6] for the case of real type potentials and was later extended by Kappeler *et al.* [17] to potentials of imaginary type. We present a self-contained proof of both theorems, which applies simultaneously to the case of real and imaginary type potentials. A clever use of *shifted w-norms* reduces the amount of technical estimates needed to a minimum and completely avoids the consideration of »slowly growing« weights. This significantly simplifies the original approaches, and allows a straightforward generalization to potentials not necessarily of real or imaginary type.

As a byproduct we obtain the following result due to Tkachenko [25] about finite gap potentials.

Theorem 3 *For each $w \in \mathcal{M}$ the class of general, real type and imaginary type finite gap potentials is dense in $\mathcal{H}^w$, $\mathcal{H}^w \cap \mathcal{L}_r^2$ and $\mathcal{H}^w \cap \mathcal{L}_i^2$, respectively.* ⋊

When the potential φ is neither assumed to be of real nor of imaginary type, the gap lengths may not encode *any* information about the regularity of the potential. Indeed, for Hill's operator with q allowed to be complex, Gasymov [8] provided an explicit construction of potentials of any regularity, whose gaps are all collapsed. Additional spectral data are needed to infer the regularity in this case. To this end, Sansuc & Tkachenko [23] considered the quantities $\delta_n = \tau_n - \mu_n$, where $\tau_n \coloneqq (\lambda_n^+ + \lambda_n^-)/2$ is the mid point of the nth spectral gap and μ_n is the nth Dirichlet eigenvalue. Later, Djakov & Mityagin [6] adapted this approach to the Zakharov-Shabat system. To further extend their results we introduce *auxiliary gap lengths* $\delta(\varphi) \coloneqq (\delta_n)_{n\in\mathbb{Z}}$, which need to satisfy

(a) δ_n is continuously differentiable,

(b) δ_n vanishes whenever $\lambda_n^+ = \lambda_n^-$ with a geometric multiplicity of two, and

(c) there are real numbers ξ_n^+ and ξ_n^- such that

$$\partial\delta_n = t_n + o(1), \qquad t_n = \frac{1}{2}\left(e^{2\pi i n(\xi_n^- + x)}, e^{-2\pi i n(\xi_n^+ + x)}\right),$$

locally uniformly in φ, where the factor $1/2$ is arbitrary but simplifies computations involving normalized quantities.

The auxiliary gap lengths $\delta(\varphi)$ complement the gap lengths $\gamma(\varphi)$ in the case of a general potential, and allow us to obtain the regularity results for the general case by the same line of arguments as used in the real or imaginary case. One easily verifies that $\delta_n = \tau_n - \mu_n$ and even $\delta_n = \tau_n - \nu_n$, with μ_n the nth Dirichlet and ν_n the nth Neumann eigenvalue, satisfies the conditions (a)-(c), which justifies the introduction of auxiliary gap lengths. Since the latter include $\delta_n = \tau_n - \mu_n$ as a special case, we can further extend the results of [6, 17].

Theorem 4 *Suppose $w \in \mathcal{M}$ and $\varphi \in \mathcal{L}^2$ is a general potential with gap lengths $\gamma(\varphi)$ and auxiliary gap lengths $\delta(\varphi)$.*

(a) *If $\varphi \in \mathcal{H}^w$, then $\gamma(\varphi) \in h^w$ and $\delta(\varphi) \in h^w$.*

(b) *Conversely, if $\gamma(\varphi) \in h^w$ and $\delta(\varphi) \in h^w$, then $\varphi \in \mathcal{H}^w$ when w is subexponential, and $\varphi \in \mathcal{H}^{w_\varepsilon}$ for some $\varepsilon > 0$ when w is exponential.* ⋊

Djakov & Mityagin [6] considered the quantities λ_n^-, $\tau_n + \delta_n$ and λ_n^+ as the vertices of a spectral triangle Δ_n and

$$\Gamma_n(\varphi) \coloneqq |\gamma_n(\varphi)| + |\delta_n(\varphi)|$$

as a measure of its size. With this notation, we can summarize our results as follows.

Theorem 5 *If w is subexponential, then*

$$\varphi \in \mathcal{H}^w \quad \Leftrightarrow \quad \Gamma(\varphi) \in h^w. \quad ⋊$$

Well-posedness

Now suppose the potential φ also depends differentiably on time t, and denote it by φ^t. In the defocusing and focusing case the gap lengths $\gamma(\varphi^t)$ completely suffice to characterize the regularity of φ^t. Moreover, the NLS flow leaves the periodic spectrum and in particular the gap lengths invariant, that is $\gamma(\varphi^t) = \gamma(\varphi^0)$. Thus, if we restrict ourselves to subexponential weights, we obtain the following implications clockwise from top left to bottom left:

$$\begin{array}{ccc} \varphi^0 \in \mathcal{H}^w & \Longrightarrow & \gamma(\varphi^0) \in h^w \\ & & \Downarrow \\ \varphi^t \in \mathcal{H}^w & \Longleftarrow & \gamma(\varphi^t) \in h^w. \end{array}$$

Consequently, any solution of the defocusing or focusing NLS equation, whose initial condition φ^0 belongs to $\mathcal{H}^w$, stays in the weighted Sobolev space $\mathcal{H}^w$ for all time, and is even uniformly bounded.

We sketch an approach to further extend this result – see also [16]. Consider the global Birkhoff coordinates $(x_n, y_n)_{n\in\mathbb{Z}}$ on $\mathcal{L}^2_r$ constructed for the defocusing NLS equation in [22]. They are the cartesian counterpart to action angle coordinates $(I_n, \theta_n)_{n\in\mathbb{Z}}$ and linearize the NLS equation to

$$\begin{aligned} \dot{x}_n &= \quad \omega_n y_n, \\ \dot{y}_n &= -\omega_n x_n, \end{aligned}$$

where ω_n only depends on the actions $I_n = 1/2(x_n^2 + y_n^2)$. Moreover,

$$|x_n|^2 + |y_n|^2 = O(|\Gamma_n|^2)$$

on some complex neighbourhood of $\mathcal{L}^2_r$ in $\mathcal{L}^2$. Using Theorem 5 one can show that for every subexponential weight w the Birkhoff map $\Omega\colon \mathcal{H}^w_r \to h^w_r \times h^w_r$, which essentially is a nonlinear Fourier transformation that introduces Birkhoff coordinates, is a real analytic diffeomorphism. Thus the regularity of any initial condition is preserved in Birkhoff coordinates as $\varphi^0 \in \mathcal{H}^w_r$ is mapped to the sequence space $h^w_r \times h^w_r$, where integrating is trivial. Consequently, the defocusing NLS equation is globally well posed in the sense of Hadamard, i.e. solutions exist for all time, are unique, and depend continuously on their initial data. This encompasses all high regularity classes generated by subexponential weights, such as Sobolev and Gevrey spaces.

The general NLS equation, including the focusing case, also admits Birkhoff coordinates, albeit, only locally around the zero solution. These are sufficient to infer the local well-posedness of the general NLS equation in weighted Sobolev spaces.

Acknowledgment. I wish to thank my supervisor Jürgen Pöschel for careful leading and valuable help with writing this thesis.

Chapter 1

The forward problem

Let $L^2 \coloneqq L^2(\mathbb{T}, \mathbb{C})$ denote the space of 1-periodic, L^2-integrable, complex functions on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We consider Zakharov-Shabat operators

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix},$$

with $\varphi = (\varphi_-, \varphi_+)$ being a vector potential taken from $\mathcal{L}^2 \coloneqq L^2 \times L^2$.

The periodic spectrum of L is defined with respect to the $\mathcal{L}^2$-dense domain of 1-periodic or 1-antiperiodic functions that are together with their derivative L^2-integrable,

$$\mathcal{D}_{\mathrm{per}} \coloneqq \left\{ f \in H^1([0,1], \mathbb{C}) \times H^1([0,1], \mathbb{C}) : f(0) = \pm f(1) \right\}.$$

To avoid distinguishing between the periodic and the anti-periodic case, we use the fact that the periodic spectrum coincides with the spectrum of L considered on $[0,2]$ endowed solely with periodic boundary conditions. Consequently, a complex number λ is a periodic eigenvalue if and only if the equation

$$Lf = \lambda f$$

admits a nontrivial solution in $\mathcal{L}^2_*$, which denotes the 2-periodic version of $\mathcal{L}^2$. As the periodic spectrum is known to be pure point, it is given by the set of all such λ. Moreover, the following result is well known [22] and will be eventually recovered in the following:

Proposition 1.1 *The periodic spectrum of $L(\varphi)$ consists of a sequence of pairs of complex eigenvalues $\lambda_n^+(\varphi)$ and $\lambda_n^-(\varphi)$, listed with multiplicities, such that*

$$\lambda_n^\pm(\varphi) = n\pi + \ell_n^2, \qquad n \in \mathbb{Z},$$

locally uniformly in φ. Here, ℓ_n^2 denotes a generic ℓ^2-sequence. ⋊

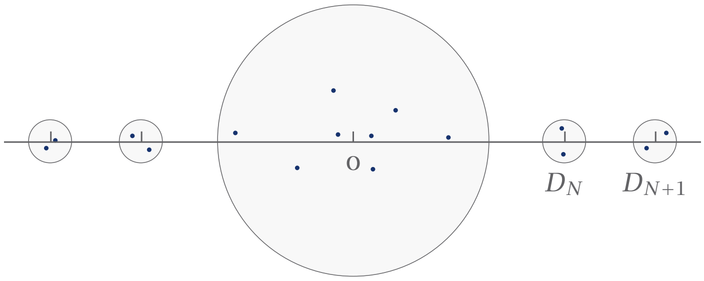

Figure 1.1: Distribution of the periodic eigenvalues.

Consequently, when $N \geqslant 1$ is chosen sufficiently large, each disc

$$D_n \coloneqq \{\lambda : |\lambda - n\pi| \leqslant \pi/6\}, \qquad |n| \geqslant N,$$

contains exactly two eigenvalues $\lambda_n^\pm$. The remaining part of the spectrum is given by an even number of periodic eigenvalues located in a bounded disc centered at the origin. If we employ a lexicographical ordering on the complex numbers by

$$a \preccurlyeq b :\Leftrightarrow \begin{cases} \operatorname{Re} a < \operatorname{Re} b \\ \qquad \text{or} \\ \operatorname{Re} a = \operatorname{Re} b \text{ and } \operatorname{Im} a \leqslant \operatorname{Im} b, \end{cases}$$

then the periodic spectrum may be represented as an unbounded bi-infinite sequence of eigenvalues with

$$\ldots \preccurlyeq \lambda_{n-1}^+ \preccurlyeq \lambda_n^- \preccurlyeq \lambda_n^+ \preccurlyeq \lambda_{n+1}^- \preccurlyeq \ldots,$$

where equality or inequality may occur in every place. While $\lambda_n^\pm$ is well defined through the lexicographical ordering for every $n \in \mathbb{Z}$, only for n sufficiently large λ_n^+ and λ_n^- may be considered as an isolated pair of eigenvalues close to $n\pi$.

Definition *The nth spectral gap length of $\varphi \in \mathcal{L}^2$ is given by*

$$\gamma_n \coloneqq \lambda_n^+ - \lambda_n^-,$$

and $\gamma(\varphi) = (\gamma_n)_{n\in\mathbb{Z}}$ denotes the entire sequences of spectral gap lengths of φ. ⋊

This chapter is dedicated to the investigation of the forward problem. That is, controlling the asymptotic behaviour of the spectral gap lengths in terms of the regularity of the potential φ. The proposition above gives a first rough answer, as $\gamma(\varphi)$ is an ℓ^2-sequence for any $\varphi \in \mathcal{L}^2$. Using the language of weighted Sobolev spaces – see appendix A for an introduction – we can give a more precise answer.

Theorem 1 *If $\varphi \in \mathcal{H}^w$ and $w \in \mathcal{M}$, then $\gamma(\varphi) \in h^w$.* ⋊

The proof of Theorem 1 is based on a Lyapunov-Schmidt reduction called *Fourier block decomposition*, which reduces the infinite-dimensional eigenvalue equation $Lf = \lambda f$ for λ close to $n\pi$ to a two-dimensional system described by a 2×2 matrix $S_n(\lambda)$. More to the point, λ is a 2-periodic eigenvalue of L if and only if $S_n(\lambda)$ is singular. In view of this correlation, the nth gap length γ_n is just the spacing of the two roots of the analytic function $\det S_n(\lambda)$, and thus may be estimated in terms of the coefficients of S_n. In turn, the asymptotic behavior of these coefficients is closely related to the regularity of φ, such that we can infer the claim. This approach was first used by Kappeler & Mityagin [14] to obtain a similar result for Hill's operator, and was later extended by Grébert & Kappeler [9, 10] to the Zakharov-Shabat system.

Our proof makes use of *shifted* w-norms, which completely avoids the delicate estimates of Neumann expansions as well as the need to introduce »slowly growing« weights and the use of iterative arguments for convolution operators implemented in [14, 17]. As a side effect, this allows a straightforward extension of the inverse problem to general potentials, as well as an improved two term gap length estimate which is presented in section 11.

2 Preparation

We start with the free equation $\varphi \equiv 0$. The 2-periodic eigenvalues of L are then exactly the double eigenvalues $k\pi$ for $k \in \mathbb{Z}$. A basis of corresponding eigenfunctions in $\mathcal{L}^2_*$, the 2-periodic version of $\mathcal{L}^2$, is given by

$$\left\{ e_k^-, e_k^+ : k \in \mathbb{Z} \right\},$$

where

$$e_k^- \coloneqq (\mathrm{e}_{-k}, 0), \quad e_k^+ \coloneqq (0, \mathrm{e}_k), \qquad \mathrm{e}_k(x) \coloneqq \mathrm{e}^{\pi i k x}.$$

This basis is orthonormal with respect to the $\mathcal{L}^2_*$ inner product

$$\langle f, g \rangle \coloneqq \langle (f_-, f_+), (g_-, g_+) \rangle = \frac{1}{2} \int_0^2 (f_- \overline{g}_- + f_+ \overline{g}_+) \mathrm{d}x,$$

and any 2-periodic vector-function f admits the following Fourier series expansion in $\mathcal{L}^2_*$,

$$f = (f_-, f_+) = \sum_{k\in\mathbb{Z}} (f_k^- \mathrm{e}_{-k}, f_k^+ \mathrm{e}_k) = \sum_{k\in\mathbb{Z}} (f_k^- e_k^- + f_k^+ e_k^+).$$

In the following we tag functions whose Fourier expansion are given in e_{-l} with the subscript $_-$ and functions whose Fourier expansion are given in e_l with the subscript $_+$. Moreover, we introduce the 2-periodic versions H^w_* and $\mathcal{H}^w_*$ of H^w and $\mathcal{H}^w$, respectively. The w-norm of a 2-periodic function $u = \sum_k u_k \mathrm{e}_k$ is given by

$$\|u\|_w^2 := \sum_{k\in\mathbb{Z}} w_k^2 |u_k|^2,$$

and the w-norm of a 2-periodic vector function $f = (f_-, f_+)$ by

$$\|f\|_w^2 := \|f_-\|_w^2 + \|f_+\|_w^2.$$

We now turn to the case of a nonzero potential φ taken from $\mathcal{H}^0$, where $0 \equiv 1$ denotes the trivial weight, that is $\mathcal{H}^0 = \mathcal{L}^2$. For $|\lambda|$ sufficiently large, the eigenvalue equation $Lf = \lambda f$ can then be viewed as a perturbation of the free equation, hence we expect solutions λ to be close to some $n\pi$. For this reason, we cover the complex plane with the closed strips

$$U_n := \{\lambda : |\mathrm{Re}\,\lambda - n\pi| \leqslant \pi/2\},$$

and consider the equation on each of these strips separately.

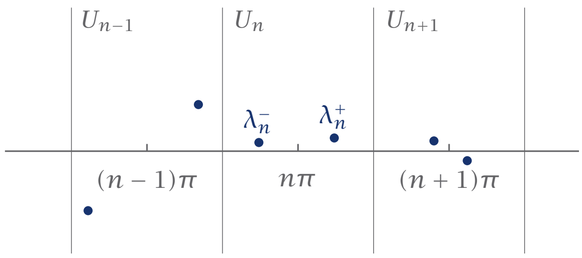

Figure 1.2: Periodic eigenvalues for sufficiently large $|n|$.

For $\lambda \in U_n$ and $|n|$ sufficiently large, the dominant modes of a solution f are e_n^+ and e_n^-. Therefore, it makes sense to separate these modes from the others by a Lyapunov-Schmidt reduction. To this end, consider the splitting

$$\begin{aligned}\mathcal{H}^0_* &= \mathcal{P}_n \oplus \mathcal{Q}_n \\ &= \mathrm{sp}\{e_n^+, e_n^-\} \oplus \overline{\mathrm{sp}}\{e_k^+, e_k^- : k \neq n\}.\end{aligned}$$

The projections onto $\mathcal{P}_n$ and $\mathcal{Q}_n$ are denoted by P_n and Q_n, respectively.

Then write the eigenvalue equation $Lf = \lambda f$ as

$$A_\lambda f = \Phi f,$$

with the unbounded operators

$$A_\lambda = \lambda - \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x}, \qquad \Phi = \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}.$$

Recall that $\varphi_\pm = \sum_{l\in\mathbb{Z}} \varphi_l^\pm \mathrm{e}_{\pm l}$ by the $_\pm$-convention, thus we get for $f = (f_-, f_+)$ in the domain of Φ,

$$\begin{aligned} \Phi f &= (\varphi_- f_+, \varphi_+ f_-) \\ &= \sum_{k,l\in\mathbb{Z}} \left(\varphi_{l+k}^- f_k^+ \mathrm{e}_{-l},\ \varphi_{l+k}^+ f_k^- \mathrm{e}_l \right) \\ &= \sum_{k,l\in\mathbb{Z}} \left(\varphi_{l+k}^- f_k^+ e_l^- + \varphi_{l+k}^+ f_k^- e_l^+ \right). \end{aligned}$$

Note that A_λ leaves the spaces $\mathcal{P}_n$ and $\mathcal{Q}_n$ invariant, hence by writing

$$f = u + v = P_n f + Q_n f,$$

we can decompose the equation $A_\lambda f = \Phi f$ into the two equations

$$\begin{aligned} A_\lambda u &= P_n \Phi(u + v), \\ A_\lambda v &= Q_n \Phi(u + v), \end{aligned}$$

called the P- and the Q-equation, respectively.

Our first goal is to solve the Q-equation on the strips U_n by writing Φv as a function of u, thus reducing the infinite dimensional eigenvalue equation $Lf = \lambda f$ to a two-dimensional system on $\mathcal{P}_n$.

Since $A_\lambda e_m^\pm = (\lambda - m\pi) e_m^\pm$, the operator A_λ is on one hand clearly unbounded, while one checks that for $m \neq n$

$$\min_{\lambda\in U_n} |\lambda - m\pi| \geqslant |n - m| \geqslant 1.$$

So, on the other hand, the restriction of A_λ to $\mathcal{Q}_n$ is boundedly invertible for all $\lambda \in U_n$. We may therefore multiply the Q-equation from the left by ΦA_λ^{-1}, giving

$$\begin{aligned} \Phi v &= \Phi A_\lambda^{-1} Q_n \Phi(u + v) \\ &= T_n \Phi(u + v), \end{aligned}$$

with $T_n \coloneqq \Phi A_\lambda^{-1} Q_n$. The latter equation may be written as

$$(\mathrm{Id} - T_n)\Phi v = T_n \Phi u.$$

Hence writing Φv as a function of u amounts to inverting $(\mathrm{Id} - T_n)$.

With foresight to section 4 we consider operator norms induced by *shifted w-norms* [21]. For any u in H_*^o the i-shifted w-norm is given by

$$\|u\|_{w;i}^2 \coloneqq \|u\mathrm{e}_i\|_w^2 = \sum_{k\in\mathbb{Z}} w_{k+i}^2 |u_k|^2, \qquad i \in \mathbb{Z}.$$

An i-shift of u shifts the Fourier coefficients to the left by i. To extend this notion to vector functions $f = (f_-, f_+) \in \mathcal{H}_*^o$, consider the Fourier series of the i-shifted components

$$f_-\mathrm{e}_{+i} = \sum_{k\in\mathbb{Z}} f_k^- \mathrm{e}_{-k+i}, \qquad f_+\mathrm{e}_{+i} = \sum_{k\in\mathbb{Z}} f_k^+ \mathrm{e}_{k+i}.$$

We observe that the Fourier coefficients of f_+ and f_- are shifted into opposite directions. To compensate this, we define the i-shifted w-norm of a vector function f by

$$\begin{aligned}
\|f\|_{w;i}^2 &\coloneqq \|f_-\|_{w;-i}^2 + \|f_+\|_{w;i}^2 = \|f_-\mathrm{e}_{-i}\|_w^2 + \|f_+\mathrm{e}_{+i}\|_w^2 \\
&= \sum_{k\in\mathbb{Z}} \left(w_{-k-i}^2 |f_k^-|^2 + w_{k+i}^2 |f_k^+|^2 \right) \\
&= \sum_{k\in\mathbb{Z}} \left(w_{k+i}^2 |f_k^-|^2 + w_{k+i}^2 |f_k^+|^2 \right).
\end{aligned}$$

Lemma 1.2 *If $\varphi \in \mathcal{H}^w$ with $w \in \mathcal{M}$, then for any $n \in \mathbb{Z}$ and any $\lambda \in U_n$,*

$$T_n = \Phi A_\lambda^{-1} Q_n$$

is a bounded linear operator on $\mathcal{H}_*^w$. Moreover, for any $i \in \mathbb{Z}$, and any $f \in \mathcal{H}_*^w$

$$\|T_n f\|_{w;i} \leqslant 2\|\varphi\|_w \|f\|_{w;-i}. \quad \rtimes$$

Proof. Given a function $f \in \mathcal{H}_*^w$, we first write $T_n f = \Phi g$ with $g = A_\lambda^{-1} Q_n f$. An estimate of the weighted L^1-norm of g by Young's inequality will then give the desired estimate of $\|T_n f\|_{w;i}$.

Recall that the restriction of A_λ to $\mathcal{Q}_n$ is boundedly invertible. Thus the function

$$g = A_\lambda^{-1} Q_n f = \sum_{m\neq n} \left(\frac{f_m^-}{\lambda - m\pi} e_m^- + \frac{f_m^+}{\lambda - m\pi} e_m^+ \right) = (g_-, g_+)$$

is well defined. With the use of Hölder's inequality we obtain for the weighted L^1-norm

$$\|g_+ \mathrm{e}_{-i}\|_{1,w} = \sum_{m\neq n} \frac{w_{m-i}|f_m^+|}{|\lambda - m\pi|} \leqslant \left(\sum_{m\neq n} \frac{1}{|n-m|^2}\right)^{1/2} \|f_+\|_{w;-i} \leqslant 2\|f_+\|_{w;-i},$$

uniformly for $\lambda \in U_n$. And, in a similar fashion, $\|g_- \mathrm{e}_i\|_{1,w} \leqslant 2\|f_-\|_{w;i}$. Note that

$$\|\Phi g\|_{w;i} = \|(\varphi_- g_+)\mathrm{e}_{-i}, (\varphi_+ g_-)\mathrm{e}_i\|_w = \|\varphi_-(g_+\mathrm{e}_{-i}), \varphi_+(g_-\mathrm{e}_i)\|_w.$$

The claim now follows with the estimate above and the weighted Young inequality B.1,

$$\begin{aligned}\|T_n f\|_{w;i}^2 &= \|\Phi g\|_{w;i}^2 \leqslant \|\varphi\|_w^2 \left(\|g_-\mathrm{e}_i\|_{1,w}^2 + \|g_+\mathrm{e}_{-i}\|_{1,w}^2\right) \\ &\leqslant 4\|\varphi\|_w^2 \left(\|f_-\|_{w;i}^2 + \|f_+\|_{w;-i}^2\right) \\ &= 4\|\varphi\|_w^2 \|f\|_{w;-i}^2. \quad \blacksquare\end{aligned}$$

Provided $\varphi \in \mathcal{H}^w$ the operator T_n is in particular bounded on $\mathcal{H}^w$ with respect to operator norm induced by the 0-shifted w-norm,

$$\|T_n\|_w \leqslant 2\|\varphi\|_w.$$

In order to achieve the bounded invertibility of $(\mathrm{Id} - T_n)$ we show that $\|T_n^2\|_0 \to 0$ as $|n| \to \infty$. Consequently, $(\mathrm{Id} - T_n^2)$ is boundedly invertible on $\mathcal{H}_*^0$, for all $|n|$ sufficiently large, using a Neumann series. This gives, in view of the identity

$$(\mathrm{Id} - T_n)^{-1} = (\mathrm{Id} + T_n)(\mathrm{Id} - T_n^2)^{-1},$$

the desired result for $(\mathrm{Id} - T_n)$.

To avoid technical complications we further assume that all weights taken from $\mathcal{M}$ are *monotone*, that is

$$w_n \leqslant w_{n+1}, \qquad n \geqslant 0.$$

The case of general, eventually non-monotone weights is discussed in section 10.

Lemma 1.3 *Suppose $\varphi \in \mathcal{H}^w$ with $w \in \mathcal{M}$. Then for any $n \in \mathbb{Z}$ and any $\lambda \in U_n$,*

$$\|T_n^2\|_{w;n} \leqslant 4\|\varphi\|_w \left(\frac{\|\varphi\|_w}{\sqrt{|n|}} + \frac{\|R_n\varphi\|_w}{w_n}\right),$$

where $R_n\varphi$ denotes the nth remainder of a function φ,

$$R_n\varphi = \sum_{|k|\geqslant n} (\varphi_k^- e_k^- + \varphi_k^+ e_k^+). \quad \rtimes$$

Proof. As in the preceding lemma, given $f = (f_-, f_+) \in \mathcal{H}_*^w$, we write $T_n^2 f = \Phi g$ with

$$g = A_\lambda^{-1} Q_n \Phi A_\lambda^{-1} Q_n f.$$

Then a straightforward computation yields

$$g \coloneqq \sum_{k,l \neq n} \left(\frac{\varphi_{k+l}^-}{\lambda - k\pi} \frac{f_l^+}{\lambda - l\pi} e_k^- + \frac{\varphi_{k+l}^+}{\lambda - k\pi} \frac{f_l^-}{\lambda - l\pi} e_k^+ \right) = (g_-, g_+).$$

To proceed, we estimate the weighted L^1-Norms $\|g_+ \mathrm{e}_{-n}\|_{1,w}$ and $\|g_- \mathrm{e}_{+n}\|_{1,w}$. An application of Lemma I.2 gives

$$\begin{aligned}
\|g_+ \mathrm{e}_{-n}\|_{1,w} &\leqslant \sum_{k,l \neq n} w_{k-n} \frac{|\varphi_{k+l}^+|}{|n-k|} \frac{|f_l^-|}{|n-l|} \\
&\leqslant 4 \left(\frac{\|\varphi_+\|_w}{\sqrt{|n|}} + \frac{\|R_n \varphi_+\|_w}{w_n} \right) \left(\sum_l w_{-l-n}^2 |f_l^-|^2 \right)^{1/2} \\
&= 4 \left(\frac{\|\varphi_+\|_w}{\sqrt{|n|}} + \frac{\|R_n \varphi_+\|_w}{w_n} \right) \|f_- \mathrm{e}_{-n}\|_w,
\end{aligned}$$

as well as,

$$\begin{aligned}
\|g_- \mathrm{e}_n\|_{1,w} &\leqslant \sum_{k,l \neq n} w_{-k+n} \frac{|\varphi_{k+l}^-|}{|n-k|} \frac{|f_l^+|}{|n-l|} \\
&\leqslant 4 \left(\frac{\|\varphi_-\|_w}{\sqrt{|n|}} + \frac{\|R_n \varphi_-\|_w}{w_n} \right) \left(\sum_l w_{l+n}^2 |f_l^+|^2 \right)^{1/2} \\
&= 4 \left(\frac{\|\varphi_-\|_w}{\sqrt{|n|}} + \frac{\|R_n \varphi_-\|_w}{w_n} \right) \|f_+ \mathrm{e}_n\|_w.
\end{aligned}$$

Since $\|T_n^2 f\|_{w;n} = \|\Phi g\|_{w;n} = \|\varphi_-(g_+ \mathrm{e}_{-n}), \varphi_+(g_- \mathrm{e}_n)\|_w$, we can use the above estimates together with the weighted Young inequality B.1 to obtain

$$\begin{aligned}
\|T_n^2 f\|_{w;n}^2 &\leqslant \|\varphi\|_w^2 \left(\|g_+ \mathrm{e}_{-n}\|_{1,w}^2 + \|g_- \mathrm{e}_n\|_{1,w}^2 \right) \\
&\leqslant 16 \|\varphi\|_w^2 \left(\frac{\|\varphi\|_w}{\sqrt{|n|}} + \frac{\|R_n \varphi\|_w}{w_n} \right)^2 \left(\|f_-\|_{w;-n}^2 + \|f_+\|_{w;n}^2 \right) \\
&= 16 \|\varphi\|_w^2 \left(\frac{\|\varphi\|_w}{\sqrt{|n|}} + \frac{\|R_n \varphi\|_w}{w_n} \right)^2 \|f\|_{w;n}^2. \qquad \blacksquare
\end{aligned}$$

It follows from Lemma 1.3 that T_n^2 is a $1/2$-contraction on $\mathcal{H}_*^0$ for all $|n|$ sufficiently large, where the threshold of $|n|$ can be chosen locally uniformly in φ on $\mathcal{H}^0$ due to the $R_n \varphi$ term. The situation becomes even more transparent, when the weight w is assumed to be unbounded, as $\mathcal{H}^w$ is then compactly embedded into $\mathcal{H}^0$ by Rellich's Theorem

C.3. Hence the contraction property holds uniformly in φ on bounded subsets of $\mathcal{H}^w$. Moreover, in view of Lemma C.1 we may always assume the unboundedness of w when $\varphi \in \mathcal{H}^w$ is given. In this case Lemma 1.3 says

$$\|T_n^2\|_0,\ \|T_n^2\|_{w;n} \leqslant \frac{8}{\hat{w}_n}\|\varphi\|_w^2, \qquad \hat{w}_n = \min\left(\sqrt{|n|}, w_n\right),$$

and the threshold for $|n|$ can explicitly be chosen uniformly on bounded subsets of $\mathcal{H}^w$.

To simplify notation, we denote the subclass of unbounded, monotone weights by $\mathcal{M}^\star$. Moreover, we introduce the weighted balls

$$\mathcal{B}_m^w \coloneqq \{\varphi \in \mathcal{H}^w : \|\varphi\|_w \leqslant m\}, \qquad m \geqslant 1,$$

which are centered at the origin. The following proposition summarizes our observations.

Proposition 1.4 *For each $w \in \mathcal{M}^\star$, and each $m \geqslant 1$ there exists an $N_{m,w} \geqslant 1$ such that*

$$\|T_n^2\|_0,\ \|T_n^2\|_{w;n} \leqslant 1/2, \qquad |n| \geqslant N_{m,w},$$

uniformly for $\lambda \in U_n$ and $\varphi \in \mathcal{B}_m^w$. ⋊

3 Reduction

With the preparation of the preceding section we can reduce the infinite dimensional eigenvalue equation $Lf = \lambda f$ to a two-dimensional system.

For the remaining part of this chapter $m \geqslant 1$ and $w \in \mathcal{M}^\star$ are considered to be fixed, and is $N_{m,w}$ is chosen according to Proposition 1.4 above. In view of the identity

$$\hat{T}_n \coloneqq (\mathrm{Id} - T_n)^{-1} = (\mathrm{Id} + T_n)(\mathrm{Id} - T_n^2)^{-1},$$

one then finds a unique solution

$$\Phi v = \hat{T}_n T_n \Phi u$$

of the Q-equation for each $|n| \geqslant N_{m,w}$. Substituting this solution into the P-equation yields

$$A_\lambda u = P_n(\mathrm{Id} + \hat{T}_n T_n)\Phi u = P_n \hat{T}_n \Phi u.$$

Writing the latter as

$$S_n u = 0, \qquad S_n \coloneqq A_\lambda - P_n \hat{T}_n \Phi,$$

we immediately conclude that there exists a one-to-one relationship between a nontrivial solution u of $S_n u = 0$ and a nontrivial 2-periodic solution f of $Lf = \lambda f$. An immediate consequence is the following lemma.

Reduction Lemma 1.5 *Suppose $|n| \geqslant N_{m,w}$, then a complex number $\lambda \in U_n$ is a periodic eigenvalue of L if and only if it is a root of the determinant of S_n.* ⋊

Recall that P_n is the orthogonal projection onto the two-dimensional space $\mathcal{P}_n$. The matrix representation of an operator B on $\mathcal{P}_n$ is given by

$$(\langle Be_n^{\pm}, e_n^{\pm}\rangle)_{\pm,\pm} .$$

Therefore, we find for S_n the representation

$$A_\lambda = \begin{pmatrix} \lambda - n\pi & \\ & \lambda - n\pi \end{pmatrix}, \qquad P_n \hat{T}_n \Phi = \begin{pmatrix} a_n^+ & b_n^+ \\ b_n^- & a_n^- \end{pmatrix},$$

with the coefficients of the latter matrix given by

$$a_n^+ \coloneqq \left\langle \hat{T}_n \Phi e_n^+, e_n^+ \right\rangle , \qquad b_n^+ \coloneqq \left\langle \hat{T}_n \Phi e_n^-, e_n^+ \right\rangle ,$$
$$b_n^- \coloneqq \left\langle \hat{T}_n \Phi e_n^+, e_n^- \right\rangle , \qquad a_n^- \coloneqq \left\langle \hat{T}_n \Phi e_n^-, e_n^- \right\rangle .$$

We point out that the coefficient functions of S_n depend on λ and φ. They share some remarkable symmetries which we will discuss in the following.

Lemma 1.6 *Let $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$. For any $\lambda \in U_n$ and any $|n| \geqslant N_{m,w}$,*

a) $a_n^+ = a_n^-$,

b) $a_n^{\pm}(\overline{\lambda}) = \overline{a_n^{\pm}(\lambda)}$ *and* $b_n^+(\overline{\lambda}) = \pm\overline{b_n^-(\lambda)}$, *when* $\varphi_+ = \pm\overline{\varphi_-}$. ⋊

Therefore, we drop the superscript of $a_n^{\pm}$ in the sequel.

Proof. a): Recall that $T_n = \Phi A_\lambda^{-1} Q_n$. From evaluating the diagonal operators A_λ and Q_n at $e_k^{\pm}$ we conclude

$$A_\lambda^* = A_{\overline{\lambda}} = P\overline{A_\lambda}P, \quad Q_n^* = P\overline{Q_n}P = Q_n,$$

where $^-$ denotes complex conjugation and

$$P = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Since A_λ leaves $\mathcal{Q}_n$ invariant, we get $(A_\lambda^{-1}Q_n)^* = P\overline{A_\lambda^{-1}Q_n}P$. Moreover, we have $\Phi^* = P\overline{\Phi}P$ such that

$$(T_n\Phi)^* = \Phi^*(A_\lambda^{-1}Q_n)^*\Phi^* = P\overline{T_n\Phi}P.$$

An inspection of the Neumann expansion of $\hat{T}_n\Phi$ yields $(\hat{T}_n\Phi)^* = P\overline{\hat{T}_n\Phi}P$. Together with $\overline{e_n^\pm} = Pe_n^\mp$ we then obtain

$$a_n^+ = \left\langle \hat{T}_n\Phi e_n^+, e_n^+\right\rangle = \left\langle e_n^+, (\hat{T}_n\Phi)^* e_n^+\right\rangle = \left\langle Pe_n^+, \overline{\hat{T}_n\Phi}Pe_n^+\right\rangle = \left\langle \hat{T}_n\Phi e_n^-, e_n^-\right\rangle = a_n^-.$$

b): Suppose $\varphi_+ = \pm\overline{\varphi_-}$, then we have $\Phi^* = \pm\Phi$ such that

$$\overline{T_n(\lambda)} = \overline{\Phi A_\lambda^{-1}Q_n} = PP\overline{\Phi}PP\overline{A_\lambda^{-1}}PP\overline{Q_n}PP = \pm PT_n(\overline{\lambda})P.$$

Using the identity $\hat{T}_n = (\mathrm{Id} + T_n)(\mathrm{Id} - T_n^2)^{-1}$, we get

$$\overline{\hat{T}_n(\lambda)} = P(\mathrm{Id} \pm T_n(\overline{\lambda}))(\mathrm{Id} - T_n^2(\overline{\lambda}))^{-1}P.$$

So it follows that

$$\begin{aligned} a_n^+(\overline{\lambda}) = & \left\langle \hat{T}_n(\overline{\lambda})\Phi e_n^+, e_n^+\right\rangle = \left\langle Pe_n^-, \overline{\hat{T}_n(\overline{\lambda})\Phi} Pe_n^-\right\rangle \\ &= \pm\left\langle e_n^-, P\overline{\hat{T}_n(\overline{\lambda})}P\Phi e_n^-\right\rangle \\ &= \pm\left\langle e_n^-, (\mathrm{Id} \pm T_n(\lambda))(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^-\right\rangle. \end{aligned}$$

Since T_n is an anti-diagonal operator, T_n^2 is a diagonal. Hence we obtain

$$\begin{aligned} a_n^+(\overline{\lambda}) &= \pm\left\langle e_n^-, (\mathrm{Id} \pm T_n(\lambda))(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^-\right\rangle \\ &= \left\langle e_n^-, T_n(\lambda)(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^-\right\rangle \\ &= \left\langle e_n^-, (\mathrm{Id} + T_n)(\lambda)(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^-\right\rangle \\ &= \overline{a_n^-(\lambda)}. \end{aligned}$$

For the case of b_n^+ we get on the other hand,

$$\begin{aligned} b_n^+(\overline{\lambda}) = & \left\langle \hat{T}_n(\overline{\lambda})\Phi e_n^-, e_n^+\right\rangle = \left\langle Pe_n^-, \overline{\hat{T}_n(\overline{\lambda})\Phi} Pe_n^+\right\rangle \\ &= \pm\left\langle e_n^-, P\overline{\hat{T}_n(\overline{\lambda})}P\Phi e_n^-\right\rangle \\ &= \pm\left\langle e_n^-, (\mathrm{Id} \pm T_n(\lambda))(\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^+\right\rangle \\ &= \pm\left\langle e_n^-, (\mathrm{Id} - T_n^2(\lambda))^{-1}\Phi e_n^+\right\rangle \\ &= \overline{b_n^-(\lambda)}. \quad \blacksquare \end{aligned}$$

The coefficients a_n and $b_n^\pm$ are given as scalar products on the two dimensional space $\mathcal{Q}_n$. In fact, their representation can be reduced from a two dimensional to a one dimensional. To this end, we write the anti-diagonal operator Φ and the diagonal operators A_λ and Q_n as

$$\Phi = \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}, \qquad A_\lambda = \begin{pmatrix} A_\lambda^- & \\ & A_\lambda^+ \end{pmatrix}, \qquad Q_n = \begin{pmatrix} Q_{-n} & \\ & Q_n \end{pmatrix},$$

where $A_\lambda^\pm \mathrm{e}_{\pm k} = (\lambda - k\pi)\mathrm{e}_{\pm k}$ and $Q_{\pm n}\mathrm{e}_{\pm k} = \delta_{nk}\mathrm{e}_{\pm k}$. Then we may write

$$T_n = \begin{pmatrix} & T_n^+ \\ T_n^- & \end{pmatrix}, \qquad T_n^\pm \coloneqq \varphi_\mp (A_\lambda^\pm)^{-1} Q_{\pm n}.$$

Evaluating the identity $\hat{T}_n = (\mathrm{Id} + T_n)(\mathrm{Id} - T_n^2)^{-1}$ using this representation of T_n, gives

$$\hat{T}_n = \begin{pmatrix} \mathrm{Id} & T_n^+ \\ T_n^- & \mathrm{Id} \end{pmatrix} \begin{pmatrix} (\mathrm{Id} - T_n^+ T_n^-)^{-1} & \\ & (\mathrm{Id} - T_n^- T_n^+)^{-1} \end{pmatrix}.$$

So it follows that

$$\begin{aligned} a_n &= \left\langle \hat{T}_n \Phi e_n^+, e_n^+ \right\rangle = \left\langle T_n (\mathrm{Id} - T_n^2)^{-1} \Phi e_n^+, e_n^+ \right\rangle \\ &= \left\langle T_n^- (\mathrm{Id} - T_n^+ T_n^-)^{-1} (\varphi_- \mathrm{e}_n), \mathrm{e}_n \right\rangle, \end{aligned}$$

as well as

$$\begin{aligned} b_n^+ &= \left\langle \hat{T}_n \Phi e_n^-, e_n^+ \right\rangle = \left\langle (\mathrm{Id} - T_n^2)^{-1} \Phi e_n^-, e_n^+ \right\rangle \\ &= \left\langle (\mathrm{Id} - T_n^- T_n^+)^{-1} (\varphi_+ \mathrm{e}_{-n}), \mathrm{e}_n \right\rangle, \\ b_n^- &= \left\langle \hat{T}_n \Phi e_n^+, e_n^- \right\rangle = \left\langle (\mathrm{Id} - T_n^2)^{-1} \Phi e_n^+, e_n^- \right\rangle \\ &= \left\langle (\mathrm{Id} - T_n^+ T_n^-)^{-1} (\varphi_- \mathrm{e}_n), \mathrm{e}_{-n} \right\rangle. \end{aligned}$$

We will make repeated use of these representations in the sequel.

4 Gap estimates

In the previous section we observed that for $|n| \geqslant N_{m,w}$ a complex number $\lambda \in U_n$ is a periodic eigenvalue of L if and only if the 2×2-matrix

$$S_n(\lambda) = \begin{pmatrix} \lambda - n\pi - a_n & -b_n^+ \\ -b_n^- & \lambda - n\pi - a_n \end{pmatrix}$$

is singular. The coefficients of S_n turn out to be analytic functions in λ, thus we can use Rouche's theorem to deduce that $\det S_n$ has exactly two roots in U_n, namely the periodic eigenvalues λ_n^+ and λ_n^-. Our goal is then to find a suitable estimate of the nth gap length $\gamma_n = \lambda_n^+ - \lambda_n^-$.

As a starting point we use the representations

$$a_n = \langle T_n^+(\mathrm{Id} - T_n^- T_n^+)^{-1}(\varphi_+ \mathrm{e}_{-n}), \mathrm{e}_{-n}\rangle,$$
$$b_n^\pm = \varphi_{2n}^\pm + \langle T_n^\mp T_n^\pm(\mathrm{Id} - T_n^\mp T_n^\pm)^{-1}(\varphi_\pm \mathrm{e}_{\mp n}), \mathrm{e}_{\pm n}\rangle,$$

which we obtain from the identity $(\mathrm{Id} - T_n^\pm T_n^\mp)^{-1} = \mathrm{Id} + T_n^\pm T_n^\mp(\mathrm{Id} - T_n^\pm T_n^\mp)^{-1}$ together with the results of Lemma 1.6. Moreover, we introduce the following notion for the sup-norm of a complex valued function

$$|f|_U \coloneqq \sup_{\lambda \in U} |f(\lambda)|.$$

Lemma 1.7 *Suppose $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$. Then for any $|n| \geqslant N_{m,w}$ the coefficients a_n and $b_n^\pm$ are analytic functions on U_n with bounds*

$$|a_n|_{U_n} \leqslant \frac{4}{\hat{w}_n^2}\|\varphi\|_w^2, \qquad |b_n^\pm - \varphi_{2n}^\pm|_{U_n} \leqslant \frac{8}{\hat{w}_n^2}\|\varphi\|_w^3. \quad \rtimes$$

Proof. Since $\|T_n^2\|_0 \leqslant 1/2$, the Neumann expansion of $(\mathrm{Id} - T_n^\pm T_n^\mp)^{-1}$ converges uniformly for $\lambda \in U_n$. Thus a_n and $b_n^\pm$ are the uniform limit of analytic functions and hence analytic on U_n.

To estimate $|a_n|$ put $u = (\mathrm{Id} - T_n^- T_n^+)^{-1}(\varphi_+ \mathrm{e}_{-n})$. As $\|T_n^2\|_{w;n} \leqslant 1/2$, we immediately get

$$\|u\|_{w;n} \leqslant \|(\mathrm{Id} - T_n^- T_n^+)^{-1}\|_{w;n}\|\varphi_+ \mathrm{e}_{-n}\|_{w;n} \leqslant 2\|\varphi_+\|_w.$$

Writing $u = \sum_{m\in\mathbb{Z}} u_m \mathrm{e}_m$ a straightforward computation yields

$$a_n = \langle T_n^+ u, \mathrm{e}_{-n}\rangle = \sum_{m\neq n} \frac{\varphi_{n+m}^-}{\lambda - m\pi} u_m.$$

Thus we can apply Hölder's inequality to obtain

$$\begin{aligned}
|a_n| &\leqslant \sum_{|n-m|>|n|} \frac{|\varphi_{n+m}^-|}{|n-m|}|u_m| + \frac{1}{w_n^2}\sum_{|n-m|\leqslant|n|} w_n|\varphi_{n+m}^-| \cdot w_n|u_m| \\
&\leqslant \frac{1}{|n|}\|\varphi_-\|_0\|u\|_0 + \frac{1}{w_n^2}\left(\sum_{|n-m|\leqslant|n|} w_{n+m}^2|\varphi_{n+m}^-|^2\right)^{1/2}\left(\sum_m w_{n+m}^2|u_m|^2\right)^{1/2} \\
&\leqslant \left(\frac{1}{|n|} + \frac{1}{w_n^2}\right)\|\varphi\|_w\|u\|_{w;n} \\
&\leqslant \frac{4}{\hat{w}_n^2}\|\varphi\|_w^2,
\end{aligned}$$

where we used the monotonicity of the weight together with the fact that $|n-m| \leqslant |n|$ implies $|n+m| \geqslant 2|n| - |n-m| \geqslant |n|$.

To proceed with b_n^+ we put $v = T_n^+(\mathrm{Id} - T_n^- T_n^+)^{-1}(\varphi_+ \mathrm{e}_{-n})$. Then Lemma 1.2 gives

$$\|v\|_{w;-n} \leqslant 2\|\varphi_-\|_w \|(\mathrm{Id} - T_n^- T_n^+)^{-1}\|_{w;n} \|\varphi_+ \mathrm{e}_{-n}\|_{w;n} \leqslant 4\|\varphi\|_w^2.$$

Writing $v = \sum_{m\in\mathbb{Z}} v_n \mathrm{e}_{-n}$ we compute

$$b_n^+ - \varphi_{2n}^+ = \langle T_n^- v, \mathrm{e}_n \rangle = \sum_{m\neq n} \frac{\varphi_{n+m}^+}{\lambda - m\pi} v_m,$$

so by the same line of arguments

$$\begin{aligned} |b_n^+ - \varphi_{2n}^+| &\leqslant \left(\frac{1}{|n|} + \frac{1}{w_n^2} \right) \|\varphi\|_w \|v\|_{w;-n} \\ &\leqslant \frac{8}{\hat{w}_n^2} \|\varphi\|_w^3. \end{aligned}$$

The claimed bound of $b_n^- - \varphi_{2n}^-$ follows in a similar fashion. ∎

The preceding lemma implies that the determinant of S_n

$$\det S_n = (\lambda - n\pi - a_n)^2 - b_n^+ b_n^-$$

is an analytic function in λ, which is close to $(\lambda - n\pi)^2$ for $|n|$ sufficiently large. We will use this fact to infer that the determinant has two roots λ close to $n\pi$.

Lemma 1.8 *Let $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$. Then there exists an $N_{m,w}^\star \geqslant 1$, such that, for any $|n| \geqslant N_{m,w}^\star$, the determinant of S_n has exactly two complex roots ξ_+, ξ_- in U_n, which are contained in the disc*

$$D_n = \{\lambda : |\lambda - \pi n| \leqslant \pi/6\},$$

and satisfy

$$|\xi_+ - \xi_-|^2 \leqslant 6|b_n^+||b_n^-|. \qquad \rtimes$$

Proof. Let $h = \lambda - n\pi - a_n$, and choose $N_{m,w}^\star \geqslant 1$ such that

$$|a_n|_{U_n},\ |b_n^\pm|_{U_n} \leqslant \frac{\pi}{24}, \qquad |n| \geqslant N_{m,w}^\star.$$

Since

$$|a_n|_{U_n} + |b_n^\pm|_{U_n} < \inf_{\lambda \in U_n \setminus D_n} |\lambda - n\pi|,$$

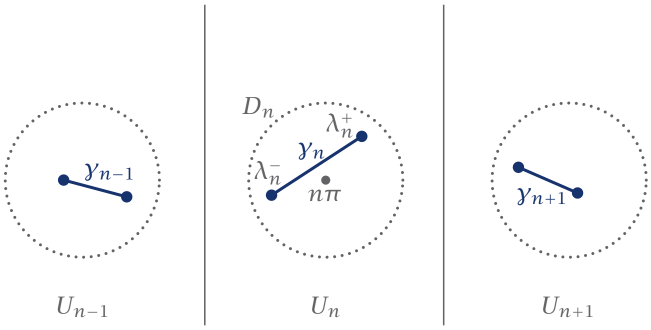

Figure 1.3: Localization of the periodic eigenvalues and their spectral gaps.

it follows from Rouche's Theorem that h has a single root in D_n, just as $(\lambda - n\pi)$. In a similar fashion, we obtain that h^2 and $\det S_n$ have the same number of roots in D_n, namely two, while $\det S_n$ clearly has no root in $U_n \setminus D_n$.

To put it differently, we write $\det S_n = g_+ g_-$ with

$$g_\pm = \lambda - \pi n - a_n \mp \sigma_n, \qquad \sigma_n = \sqrt{b_n^+ b_n^-},$$

where the branch of the root is immaterial. Then each root ξ of $\det S_n$ is either a root of g_+ or g_-, respectively, and thus satisfies $\xi = \pi n + a_n(\xi) \pm \sigma_n(\xi)$. To estimate the distance of the two roots, we note that a_n is analytic on U_n, so Cauchy's estimate F.1 gives

$$|\partial_\lambda a_n|_{D_n} \leqslant \frac{|a_n|_{U_n}}{\operatorname{dist}(D_n, \partial U_n)} \leqslant \frac{\pi/24}{\pi/3} = \frac{1}{8}.$$

The claim now follows with

$$\begin{aligned} |\xi_+ - \xi_-| &\leqslant |a_n(\xi_+) - a_n(\xi_-)| + |\sigma_n(\xi_+) \pm \sigma_n(\xi_-)| \\ &\leqslant \frac{1}{8}|\xi_+ - \xi_-| + 2|\sigma_n|_{U_n}. \qquad \blacksquare \end{aligned}$$

For $|n|$ sufficiently large, $\det S_n$ has exactly two roots on each strip U_n, counted with multiplicities. Since the strips cover the complex plane, those roots have to be the 2-periodic eigenvalues $\lambda_n^\pm$ according to the Reduction Lemma 1.5. So we obtain the following preliminary individual gap estimate

$$|\gamma_n(\varphi)|^2 = |\xi_n^+ - \xi_n^-|^2 \leqslant 6|b_n^+||b_n^-| \leqslant 3(|b_n^+|^2 + |b_n^-|^2).$$

Since $|b_n^\pm| \leqslant |\varphi_{2n}^\pm| + |b_n^\pm - \varphi_{2n}^\pm|$, it remains to show that $b_n^\pm$ is h^w-close to $\varphi_{2n}^\pm$, to conclude that $\gamma(\varphi) \in h^w$ when $\varphi \in \mathcal{H}^w$.

Proposition 1.9 *If $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$, then for any $N \geqslant N_{m,w}$*

$$\sum_{|n|\geqslant N} w_{2n}^2 |b_n^\pm - \varphi_{2n}^\pm|^2 \leqslant \frac{2^{10}}{\hat{w}_N^2} \|\varphi\|_w^6. \quad \rtimes$$

Proof. Making use of the representation of the coefficient b_n^- we obtain

$$b_n^- - \varphi_{2n}^- = \left\langle T_n^+ T_n^- (\mathrm{Id} - T_n^+ T_n^-)^{-1} (\varphi_- \mathrm{e}_n), \mathrm{e}_{-n} \right\rangle.$$

To allow an invocation of Lemma I.3, put $u_n = (\mathrm{Id} - T_n^+ T_n^-)^{-1}(\varphi_- \mathrm{e}_n)$. Then (u_n) is a sequence of H_*^w functions, that is uniformly bounded by

$$\|u_n\|_{w;-n} \leqslant \|(\mathrm{Id} - T_n^+ T_n^-)^{-1}\|_{w;-n} \|\varphi_- \mathrm{e}_n\|_{w;-n} \leqslant 2\|\varphi_-\|_w, \qquad |n| \geqslant N_{m,w}.$$

Expanding u_n into its Fourier series, $u_n = \sum_k u_{k;n} \mathrm{e}_{-k}$, we obtain by a straightforward computation

$$|\langle T_n^+ T_n^- u_n, \mathrm{e}_{-n} \rangle| = \left| \sum_{k,l \neq n} \frac{\varphi_{n+l}^-}{\lambda - l\pi} \frac{\varphi_{l+k}^+}{\lambda - k\pi} u_{k;n} \right| \leqslant \sum_{k,l\neq n} \frac{|\varphi_{n+l}^-|}{|n-l|} \frac{|\varphi_{l+k}^+|}{|n-k|} |u_{k;n}|.$$

Notice that

$$\|u_n\|_{w;-n}^2 = \sum_{k\in\mathbb{Z}} w_{k+n}^2 |u_{k;n}|^2,$$

i.e. the negative shift of u_n by $-n$ translates into a positive shift of its Fourier coefficients. With that we can invoke Lemma I.3 to obtain

$$\sum_{|n|\geqslant N} w_{2n}^2 |\langle T_n^+ T_n^- u_n, \mathrm{e}_{-n}\rangle|^2 \leqslant \frac{2^{10}}{\hat{w}_N^2} \|\varphi\|_v^6.$$

The case of b_n^+ can be treated in a similar fashion. ∎

We are now in a position to prove Theorem 1. Suppose $\varphi \in \mathcal{B}_m^w$ for some $m \geqslant 1$. Then we can choose $|n| \geqslant \max(N_{m,w}, N_{m,w}^\star)$, such that Lemma 1.8 applies giving us two roots ξ_+ and ξ_- of $\det S_n$ contained in $D_n \subset U_n$ and satisfying

$$|\gamma_n(\varphi)|^2 = |\xi_+ - \xi_-|^2 \leqslant 6|b_n^+ b_n^-|_{U_n} \leqslant 3(|b_n^+|_{U_n}^2 + |b_n^-|_{U_n}^2).$$

Moreover, one easily checks that

$$\frac{1}{2} w_{2n}^2 |b_n^\pm|^2 \leqslant w_{2n}^2 |\varphi_{2n}^\pm|^2 + w_{2n}^2 |b_n^\pm - \varphi_{2n}^\pm|^2.$$

Now, it follows from the preceding proposition that $\gamma(\varphi) \in h^w$.

Chapter 2

The inverse problem

So far we have studied the forward problem of controlling the asymptotic behaviour of the gap lengths $\gamma(\varphi)$ in terms of the regularity of the potential φ. Using the language of weighted Sobolev spaces, our result may be summarized as

$$\varphi \in \mathcal{H}^w \;\Rightarrow\; \gamma(\varphi) \in h^w.$$

We now turn to the inverse problem of recovering the regularity of the potential φ from the spectral gap lengths $\gamma(\varphi)$. Gasymov [8] observed that the gap lengths of Hill's operator $\partial_x^2 + q$ may not contain any information about the regularity of the potential q, when q is allowed to be complex. A similar observation can be made in our situation. Consider the potential

$$\eta \coloneqq \sum_{m\geqslant 1} (q_m \mathrm{e}_{2m}, p_m \mathrm{e}_{2m}).$$

Then $(\mathrm{Id} - T_n^+ T_n^-)^{-1}(\eta_- \mathrm{e}_n)$ is given as a power series in $\mathrm{e}^{2\pi i x}$ with lowest term e_{n+2}. It follows that $a_n = b_n^- = 0$ and

$$S_n = \begin{pmatrix} \lambda - n\pi & b_n^+ \\ 0 & \lambda - n\pi \end{pmatrix}.$$

So the periodic spectrum of η consists entirely of double periodic eigenvalues $n\pi$, all gaps are collapsed, and hence $\gamma(\eta) \in h^w$ for every weight w. However, the coefficients q_m and p_m can be chosen freely, such that η may have any regularity, thus $\eta \in \mathcal{H}^w$ does not hold for every w.

To avoid this situation, we first restrict ourselves to certain subclasses of $\mathcal{H}^0$-potentials. The space of potentials of *real type* is defined by

$$\mathcal{H}_r^0 \coloneqq \{\varphi \in \mathcal{H}^0 : \varphi^* = \varphi\}, \qquad \varphi^* = (\overline{\varphi_+}, \overline{\varphi_-}).$$

When φ is of real type, the operator L is formally self-adjoint. Hence the periodic spectrum of L is real, and so are the gap lengths.

The imaginary counterpart of $\mathcal{H}_r^0$ is the space of potentials of *imaginary type*,

$$\mathcal{H}_i^0 \coloneqq \{\varphi \in \mathcal{H}^0 : \varphi^* = -\varphi\} = i\mathcal{H}_r^0.$$

For φ of imaginary type, the operator L is in general not formally self-adjoint. However, each pair of eigenvalues is adjoint, that is $\lambda_n^+ = \overline{\lambda_n^-}$, such that all gap lengths are purely imaginary.

Note that $\mathcal{H}_r^0$ and $\mathcal{H}_i^0$ are *real* subspaces of $\mathcal{H}^0$, not complex ones. Moreover, the potential η given above is obviously neither of real nor of imaginary type. Indeed, the requirement for the potential to be of real or imaginary type imposes a certain symmetry on the Fourier coefficients linking them to the periodic spectrum. As a consequence the gap lengths completely encode the regularity of φ in the case of subexponential weights.

Still, for exponential weights the situation is not as clear cut as in the forward case, since there exist certain *finite gap potentials*, potentials with finitely many open gaps, that are real analytic but whose extension to the complex plane has poles. For example, one can use Weierstrass $\wp$-functions to construct a meromorphic potential $\wp_\star$ with a single gap [24]. On one hand, $\gamma(\wp_\star) \in h^w$ for any weight w, since except for one all gaps are collapsed. On the other hand, $\wp_\star$ is meromorphic with poles in the complex plane. Let $d = \min_{z\in\mathcal{P}(\wp_\star)} |\operatorname{Im} z|$ denote the shortest possible distance of a pole to the real axis. Then $\wp_\star \in \mathcal{H}^{w_a}$ with $w_a = \mathrm{e}^{a|\cdot|}$ if and only if $(2\pi)a \leqslant d$. Thus we can not expect a true converse to the forward problem for exponential weights.

Theorem 2 *Suppose $\varphi \in \mathcal{H}^0$ is of real or imaginary type and $\gamma(\varphi) \in h^w$.*

(a) *If w is subexponential, then $\varphi \in \mathcal{H}^w$.*

(b) *If w is exponential, then $\varphi \in \mathcal{H}^{w_\varepsilon}$ with $w_\varepsilon = \mathrm{e}^{\varepsilon|\cdot|}$ for all sufficiently small $\varepsilon > 0$.* ⋊

The proof of Theorem 2 is based on two observations. First, the Fourier coefficients $\varphi_{2n}^\pm$ of the potential are h^w-close to the off-diagonal coefficients $b_n^\pm$ of the matrix S_n. Second, for potentials of real or imaginary type, the latter coefficients can be bounded in terms of the nth gap length. Pöschel [20] used these facts in the similar Hill case to define a perturbed Fourier series map

$$\psi = \mathcal{F}(\varphi),$$

which, according to the inverse function theorem, is a near identity diffeomorphism that respects regularity. The Fourier coefficients of ψ are particularly suited to be compared to the gap lengths, allowing the following conclusion

$$\gamma(\varphi) \in h^w \Rightarrow \psi \in \mathcal{H}^w \Rightarrow \varphi \in \mathcal{H}^w.$$

This approach was extended by Kappeler *et al.* [17] to Zakharov-Shabat systems with potentials of real or imaginary type. Due to the special symmetries in this case, the Fourier coefficients of ψ can be bounded by the spectral gap lengths $\gamma(\varphi)$, which establishes Theorem 2.

Incidentally, we may recover an observation due to Tkachenko [25] about finite gap potentials.

Theorem 3 *For each $w \in \mathcal{M}$ the class of general, real type and imaginary type finite gap potentials is dense in $\mathcal{H}^w$, $\mathcal{H}_r^w$ and $\mathcal{H}_i^w$, respectively.* ⋊

We now turn to the inverse problem for general potentials $\varphi \in \mathcal{H}^0$. By Gasymov's observation, additional spectral data are needed to recover the regularity of the potential. For this reason, Sansuc & Tkachenko [23] in Hill's case, and later Djavok & Mityagin [6] for Zakharov-Shabat operators, considered the quantities $\delta_n = \mu_n - \tau_n$, where μ_n denotes the nth Dirichlet eigenvalue and τ_n the mid-point of the nth spectral gap. To further simplify and extend their results, we follow Pöschel's treatment of Hill's operator [21] and define an abstract set of auxiliary spectral quantities.

Definition *Suppose $\varphi \in \mathcal{H}^w$ with $w \in \mathcal{M}$. A family of auxiliary gap lengths $\delta(\varphi)$ is a bi-infinite sequence with*

(a) *δ_n is continuously differentiable on some neighbourhood $U \subset \mathcal{H}^w$ of φ,*

(b) *δ_n vanishes whenever $\lambda_n^+ = \lambda_n^-$ has also a geometric multiplicity of two, and*

(c) *there exist real numbers ξ_n^+, ξ_n^- and an integer $N_U \geqslant 1$, such that*

$$\|\partial_\varphi \delta_n - t_n\|_0 \leqslant 1/16, \qquad t_n = \frac{1}{2}\left(\mathrm{e}^{2\pi i n(\xi_n^- + \cdot)}, \mathrm{e}^{-2\pi i n(\xi_n^+ + \cdot)}\right),$$

uniformly on U for $|n| \geqslant N_U$. ⋊

For example, one can choose $\delta_n = \kappa_n - \tau_n$, with κ_n being the nth Dirichlet or Neumann eigenvalue of φ, and $\tau_n = (\lambda_n^+ - \lambda_n^-)/2$ the mid point of the nth gap – see also appendix G for the definition of Dirichlet and Neumann spectra. The factor $1/2$ in the definition of t_n is due to the usual normalization of the Dirichlet and Neumann eigenfunctions used to express the gradient $\partial\delta_n$ in this case. We show in appendix H that this choice of δ_n actually satisfies the conditions (a) - (c) whenever w is unbounded. This justifies the introduction of auxiliary gap lengths, which cover $\delta_n = \mu_n - \tau_n$ as a special case.

Similar to the case of the spectral gap lengths $\gamma(\varphi)$, the asymptotic behaviour of $\delta(\varphi)$ is determined by the regularity of the potential φ. On the other hand, no additional theory is required to recover the regularity of a general potential φ from the gap lengths complemented by a family of auxiliary gap lengths. Essentially, it is property (a) - (c) that we need to bound the Fourier coefficients of $\psi = \mathcal{F}(\varphi)$ by the auxiliary gap lengths, so we obtain the following inverse result without any additional cost.

Theorem 4 *Suppose $w \in \mathcal{M}$, $\varphi \in \mathcal{H}^0$ and $\delta(\varphi)$ is a family of auxiliary gap lengths.*

(a) *If $\varphi \in \mathcal{H}^w$, then $\gamma(\varphi) \in h^w$ and $\delta(\varphi) \in h^w$.*

(b) *Conversely, if $\gamma(\varphi) \in h^w$ and $\delta(\varphi) \in h^w$, then $\varphi \in \mathcal{H}^w$ when w is subexponential, and $\varphi \in \mathcal{H}^{w_\varepsilon}$ for some $\varepsilon > 0$ if w is exponential.* ⋊

5 Adapted Fourier coefficients

In this section we define an analytic perturbation of a given potential φ, which adapts the Fourier coefficients of φ to the spectral gap lengths while preserving regularity. The key idea is to consider the coefficients of the 2×2-matrix S_n as analytic functions of their potential and to use them as the new Fourier coefficients for the perturbed potential.

For simplicity, we assume that $\varphi \in \mathcal{H}^w$ with some unbounded and monotone weight $w \in \mathcal{M}^\star$ and not only $w \in \mathcal{M}$. In this case, $\mathcal{H}^w$ is compactly embedded into $\mathcal{L}^2$ by Rellich's Theorem, and consequently the coefficients of S_n are well defined uniformly on each ball $\mathcal{B}_m^w$ when $|n|$ is sufficiently large. In the context of regularity preservation, we need some notation to say when a weight v is "stronger" than a weight w. For this reason, we introduce the ordering

$$v \geqslant w \quad :\Leftrightarrow \quad v_n \geqslant w_n \text{ for all } n.$$

Clearly, any weight $w \in \mathcal{M}$ satisfies $w \geqslant o$ where $o \equiv 1$ denotes the trivial weight, and $\mathcal{B}_m^v \subset \mathcal{B}_m^w$ whenever $w \leqslant v$.

Lemma 2.1 *Let $w \in \mathcal{M}^\star$. One can choose $N_{m,w} \geqslant 1$, such that for any weight $v \geqslant w$*

$$\|T_n^2\|_{v;n}\Big|_{\mathcal{B}_{2m}^v} \leqslant \frac{1}{2}, \qquad |n| \geqslant N_{m,w}.$$

Moreover, the coefficient functions

$$a_n \colon U_n \times \mathcal{B}_{2m}^w \to \mathbb{C}, \qquad b_n^\pm \colon U_n \times \mathcal{B}_{2m}^w \to \mathbb{C},$$

are analytic in λ and φ, and uniformly bounded,

$$|a_n|_{U_n\times\mathcal{B}_{2m}^w},\ |b_n^\pm|_{U_n\times\mathcal{B}_{2m}^w} \leqslant \pi/48. \quad \rtimes$$

Proof. To estimate the norm of T_n^2 we apply Lemma 1.3 on the ball $\mathcal{B}_m^v$, which gives

$$\|T_n^2\|_{v;n}\big|_{\mathcal{B}_m^v} \leqslant \frac{8m^2}{\hat{v}_n} \leqslant \frac{8m^2}{\hat{w}_n}.$$

Since w is assumed to be unbounded, we can choose $N_{m,w} \geqslant 1$ such that

$$\frac{1}{\hat{w}_n} \leqslant \frac{1}{2^{12}m^2},$$

where the choice 2^{12} turns out to be useful later. Then we have $\|T_n^2\|_{v;n}\big|\mathcal{B}_{2m}^v \leqslant 1/2$ even on the ball of radius $2m$ and for any weight $v \geqslant w$. In particular, T_n^2 is a $1/2$-contraction for each $\varphi \in \mathcal{B}^w_{2m}$, hence the coefficients of S_n are well defined on $\mathcal{B}^w_{2m}$. The analytic dependence on λ and the estimates

$$|a_n|_{U_n\times\mathcal{B}^w_m} \leqslant \frac{4m^2}{\hat{w}_n^2}, \qquad |b_n^\pm|_{U_n\times\mathcal{B}^w_m} \leqslant |\varphi^\pm_{2n} + b_n^\pm - \varphi^\pm_{2n}|_{U_n\times\mathcal{B}^w_m} \leqslant \frac{m}{w_{2n}} + \frac{8m^3}{\hat{w}_n^2}.$$

follow from Lemma 1.7. Moreover, the functions a_n and $b_n^\pm$ are each given as a Neumann series in T_n^2, which converges absolutely and uniformly on $\mathcal{B}^w_{2m}$ by the contraction property of T_n^2. Hence, Lemma F.2 gives the analytic dependence on the potential φ. ∎

The idea is to perturb φ using the analytic coefficients $b_n^\pm$ of S_n as they are on one hand close to the Fourier coefficients, while they are on the other hand comparable to the gap lengths. To remove the dependency of S_n on λ, we fix a $\lambda \in U_n$ for each given φ and n. More to the point, we choose $\lambda = \alpha_n(\varphi)$ such that the diagonal of S_n vanishes, thus greatly simplifying all terms involving S_n.

Throughout this chapter $N_{m,w}$ is supposed to be chosen according to the lemma above.

Lemma 2.2 *Let $w \in \mathcal{M}^\star$. For any $|n| \geqslant N_{m,w}$ there exists a unique analytic function $\alpha_n\colon \mathcal{B}^w_{2m} \to \mathbb{C}$ such that,*

a) $|\alpha_n - n\pi|_{\mathcal{B}^w_{2m}} \leqslant \pi/48$,

b) *$\alpha_n(\varphi)$ is real for any φ of real or imaginary type, and*

c) $\alpha_n - n\pi - a_n(\alpha_n,\cdot)\Big|_{\mathcal{B}^w_{2m}} \equiv 0.$ ⋊

Proof. Let E denote the space of analytic functions $\alpha\colon \mathcal{B}^w_{2m} \to \mathbb{C}$ equipped with the usual metric induced by the topology of uniform convergence. For $|n| \geqslant N_{m,w}$, the claim can then be written as a fixed point problem of the map

$$T\alpha := n\pi + a_n(\alpha,\cdot)$$

on the closed subspace $E^\star$ of functions in E with properties a) and b). Note that $\alpha \equiv n\pi$ satisfies a) and b), hence $E^\star$ is not empty.

Each α in $E^\star$ maps the ball $\mathcal{B}^w_{2m}$ into the disc $D_n' = \{\lambda : |\lambda - n\pi| \leqslant \pi/48\} \subset U_n$ by assumption. Consequently,

$$|T\alpha - n\pi|_{\mathcal{B}^w_{2m}} = |a_n(\alpha,\cdot)|_{\mathcal{B}^w_{2m}} \leqslant |a_n|_{U_n\times\mathcal{B}^w_{2m}} \leqslant \pi/48,$$

hence $T\alpha$ satisfies a). Moreover, for φ of real or imaginary type, $a_n(\lambda,\varphi)$ is real for real λ by Lemma 1.6. Therefore, $T\alpha$ also satisfies b) and thus T maps $E^\star$ into $E^\star$.

The contraction property now follows from

$$d(T\alpha_1, T\alpha_2) = |T\alpha_1 - T\alpha_2|_{\mathcal{B}^w_{2m}} \leqslant |\partial_\lambda a_n|_{D'_n\times\mathcal{B}^w_{2m}} |\alpha_1 - \alpha_2|_{\mathcal{B}^w_{2m}},$$

and Cauchy's estimate

$$|\partial_\lambda a_n|_{D'_n\times B^w_{2m}} \leqslant \frac{|a_n|_{U_n\times B^w_{2m}}}{\operatorname{dist}(D'_n, \partial U_n)} \leqslant \frac{\pi/48}{\pi/6} = \frac{1}{8}.$$

Hence we find a unique fixed point α_n with the properties as claimed. ∎

We now define the perturbation map $\mathcal{F}_m$ on $\mathcal{B}^w_m$ by

$$\mathcal{F}_m(\varphi) := \sum_{|n|<N_{m,w}} (\varphi^-_{2n} e^-_{2n} + \varphi^+_{2n} e^+_{2n}) + \sum_{|n|\geqslant N_{m,w}} (b^-_n(\alpha_n, \cdot) e^-_{2n} + b^+_n(\alpha_n, \cdot) e^+_{2n})\Big|_\varphi.$$

For each $|n| \geqslant N_{m,w}$ the Fourier coefficients of the 1-periodic function $\psi = \mathcal{F}_m(\varphi)$ are given by $\psi^\pm_{2n} = b^\pm_n(\alpha_n, \varphi)$, and

$$S_n(\alpha_n, \varphi) = \begin{pmatrix} 0 & \psi^+_{2n} \\ \psi^-_{2n} & 0 \end{pmatrix},$$

since the diagonal of $S_n(\alpha_n, \cdot)$ vanishes. These new Fourier coefficients are adapted to the gap lengths, thus we call $\mathcal{F}_m$ *adapted Fourier coefficient map*. Indeed, for φ of real or imaginary type we make the following conclusion in section 8

$$\gamma(\varphi) \in h^w \Rightarrow \psi = \mathcal{F}_m(\varphi) \in \mathcal{H}^w.$$

With the help of some auxiliary gap lengths an analogous result for general potentials is obtained in section 9.

Next we obtain the diffeomorphism property of $\mathcal{F}_m$, which finally enables us to conclude that φ has the same regularity as ψ – with possibly a slight loss of regularity in the exponential case.

Proposition 2.3 *If $w \in \mathcal{M}^\star$, then $\mathcal{F}_m$ maps $\mathcal{B}^w_m$ into $\mathcal{H}^w$. For any weight $v \geqslant w$ its restriction to $\mathcal{B}^v_m$ is an analytic diffeomorphism*

$$\mathcal{F}_m\big|\mathcal{B}^v_m : \mathcal{B}^v_m \to \mathcal{F}_m(\mathcal{B}^v_m) \subset \mathcal{H}^v,$$

such that

$$\frac{1}{2}\|\varphi\|_v \leqslant \|\mathcal{F}_m(\varphi)\|_v \leqslant 2\|\varphi\|_v,$$

and $\mathcal{B}^v_{m/2} \subset \mathcal{F}_m(\mathcal{B}^v_m)$. Moreover, $\|\mathrm{d}\mathcal{F}_m - \mathrm{Id}\|_{\mathcal{B}^v_m} \leqslant 1/8$ and $\mathcal{F}_m$ preserves type, i.e.

$$\mathcal{F}_m(\mathcal{B}^v_m \cap \mathcal{H}^0_r) \subset \mathcal{F}_m(\mathcal{B}^v_m) \cap \mathcal{H}^0_r, \qquad \mathcal{F}_m(\mathcal{B}^v_m \cap \mathcal{H}^0_i) \subset \mathcal{F}_m(\mathcal{B}^v_m) \cap \mathcal{H}^0_i. \quad \rtimes$$

Proof. Since α_n maps the ball $\mathcal{B}_{2m}^w$ into U_n for every $|n| \geqslant N_{m,w}$, the coefficient functions $b_n^{\pm}(\alpha_n, \cdot)$ are well defined on $\mathcal{B}_{2m}^w$. Moreover, for each $v \geqslant w$ we have $\|T_n^2\|_{v;n} \leqslant 1/2$ on $\mathcal{B}_{2m}^v \subset \mathcal{B}_{2m}^w$. Thus Proposition 1.9 applies giving the estimate

$$\|\mathcal{F}_m - \mathrm{id}\|_{v,\mathcal{B}_{2m}^v}^2 = \sup_{\varphi \in \mathcal{B}_{2m}^v} \sum_{|n| \geqslant N_{m,w}} v_{2n}^2 \left(|b_n^-(\alpha_n, \cdot) - \varphi_{2n}^-|^2 + |b_n^+(\alpha_n, \cdot) - \varphi_{2n}^+|^2 \right)$$
$$\leqslant \frac{2^{18} m^6}{\hat{w}_{N_{m,w}}^2} \leqslant \frac{(2m)^2}{16^2},$$

by our choice of $N_{m,w}$ from Lemma 2.1. Therefore, $\mathcal{F}_m$ is locally bounded while each coordinate function is analytic in φ. Hence $\mathcal{F}_m$ is analytic on $\mathcal{B}_{2m}^v$ by Lemma F.3. In particular, Cauchy's estimate applies, giving

$$\|\mathrm{d}\mathcal{F}_m - \mathrm{Id}\|_{v,\mathcal{B}_m^v} \leqslant \frac{\|\mathcal{F}_m - \mathrm{id}\|_{v,\mathcal{B}_{2m}}}{\mathrm{dist}(\mathcal{B}_m, \partial\mathcal{B}_{2m})} \leqslant \frac{1}{8}.$$

The preservation of $\mathcal{H}_r^0$ and $\mathcal{H}_i^0$ follows from the symmetry of the coefficients $b_n^{\pm}$ observed in Lemma 1.6. Finally, the inverse function theorem F.4 together with the fact that $\mathcal{F}_m(0) = 0$ gives the remaining properties. ∎

We now prove Theorem 3. Suppose $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$. When $\psi_{2n}^+ = \psi_{2n}^- = 0$ for some $|n| \geqslant N_{m,w}$, then S_n vanishes identically at α_n and hence the nth gap is collapsed. Consequently, if

$$\mathcal{F}_m(\varphi) \in \mathcal{G}_N^r = \mathrm{span}\,\{(\mathrm{e}_{-k}, e_k) : |k| \leqslant N\},$$

with N sufficiently large, then at most $2N + 1$ gaps of φ are open. As the union of the spaces $\mathcal{G}_N^r$ is dense in $\mathcal{B}_m^w \cap \mathcal{H}_r^0$ and $\mathcal{F}_m$ is a diffeomorphism, it follows that the finite gap potentials are dense in $\mathcal{B}_m^w \cap \mathcal{H}_r^0$. The remaining cases are treated similarly, so the theorem is proved for unbounded weights. Moreover, $\mathcal{B}_m^w$ is dense in $\mathcal{B}_m^0$ for any weight w, so the claim immediately follows for bounded weights. The remaining cases of imaginary and general type potentials are proved analogously.

6 Abstract regularity results

What can we say about the regularity of φ, when $\psi = \mathcal{F}_m(\varphi) \in \mathcal{H}^w$ is known?

Proposition 2.4 *If $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$ and*

$$\mathcal{F}_m(\varphi) \in \mathcal{B}_{m/2}^v,$$

for a weight $v \geqslant w$ from $\mathcal{M}$, then $\varphi \in \mathcal{B}_m^v \subset \mathcal{H}^v$. ⋊

Proof. Since the weight w is assumed to be unbounded, Proposition 2.3 applies and hence $\mathcal{F}_m\colon \mathcal{B}_m^w \to \mathcal{H}^w$ is an analytic diffeomorphism onto its image. Moreover, $v \geqslant w$ by assumption, thus the restriction of $\mathcal{F}_m$ to $\mathcal{B}_m^v$ is also an analytic diffeomorphism onto its image with $\mathcal{B}_{m/2}^v \subset \mathcal{F}_m(\mathcal{B}_m^v)$. Therefore, the unique preimage φ of $\mathcal{F}_m(\varphi) \in \mathcal{B}_{m/2}^v$ has to be in $\mathcal{B}_m^v$. ∎

In general, we have no *a priori* bound on $\|\mathcal{F}_m(\varphi)\|_v$. However, we can modify the weight v to obtain such a bound while not affecting the asymptotic behaviour in the subexponential case. More to the point, for $v_\varepsilon \coloneqq \mathrm{e}^{\varepsilon|\cdot|}$ and any $\varepsilon > 0$ sufficiently small,

$$w_\varepsilon = \min(v, v_\varepsilon) \in \mathcal{M}$$

by the Cut-Off lemma D.1. If v is subexponential, then w_ε and v share the same asymptotics, while $w_\varepsilon \equiv v_\varepsilon$ for all sufficiently small $\varepsilon > 0$, when v is exponential.

It makes sense to modify only that part of the weight $v \geqslant w$, which is actually stronger than w, while leaving w untouched, as $\varphi \in \mathcal{H}^w$ is granted anyway. To this end, we introduce the notion of product weights

$$(wv)_n \coloneqq w_n \cdot v_n.$$

Obviously wv is submultiplicative if w and v are and $w \leqslant wv$ for any weight $v \in \mathcal{M}$.

Proposition 2.5 *Suppose $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$, and*

$$\mathcal{F}_m(\varphi) \in \mathcal{H}^{vw}$$

for some $v \in \mathcal{M}$.

(a) *If v is subexponential, then also $\varphi \in \mathcal{H}^{vw}$.*

(b) *If v is exponential, then $\varphi \in \mathcal{H}^{v_\varepsilon w}$ for all sufficiently small $\varepsilon > 0$.* ⋊

Proof. We may assume that $m \geqslant 8\|\varphi\|_w$, as increasing m does not change our assumptions. Put $\psi = \mathcal{F}_m(\varphi)$, then $\|\psi\|_{vw} < \infty$ by assumption, and hence $\|R_N\psi\|_{vw} \leqslant \|\psi\|_w$ for some $N \geqslant 1$. With $w_\varepsilon = \min(v, v_\varepsilon)$ we then obtain

$$\begin{aligned}
\|\psi\|_{w_\varepsilon w}^2 &= \|\psi - R_N\psi\|_{w_\varepsilon w}^2 + \|R_N\psi\|_{w_\varepsilon w}^2 \\
&\leqslant \|\psi - R_N\psi\|_{v_\varepsilon w}^2 + \|R_N\psi\|_{vw}^2 \\
&\leqslant \mathrm{e}^{\varepsilon N}\|\psi - R_N\psi\|_w^2 + \|\psi\|_w^2 \\
&\leqslant (1 + \mathrm{e}^{\varepsilon N})\|\psi\|_w^2 \\
&\leqslant 4\|\psi\|_w^2,
\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Thus, the preceding lines together with Proposition 2.3 yield

$$\|\psi\|_{w_\varepsilon w} \leqslant 2\|\psi\|_w \leqslant 4\|\varphi\|_w \leqslant m/2,$$

such that $\psi \in \mathcal{B}^{w_\varepsilon w}_{m/2}$. Since $w_\varepsilon w \geqslant w$, it follows from Proposition 2.4 that $\varphi \in \mathcal{B}^{w_\varepsilon w}_m$. Moreover, $\mathcal{H}^{w_\varepsilon w} = \mathcal{H}^{\nu w}$, provided ν is subexponential, while we have $w_\varepsilon = \nu_\varepsilon$ for any exponential ν when $\varepsilon > 0$ is sufficiently small. ∎

7 Lower gap length bound

Since $\mathcal{F}_m$ respects regularity, recovering the regularity of the potential φ from the asymptotic behaviour of its gap lengths $\gamma(\varphi)$ amounts to dominating the Fourier coefficients of $\psi = \mathcal{F}_m(\varphi)$ by γ_n, such that

$$\gamma(\varphi) \in h^\nu \quad \Rightarrow \quad \psi \in \mathcal{H}^\nu.$$

By Gasymov's observation this cannot hold for arbitrary potentials, thus we impose additional conditions on ψ.

Lemma 2.6 *Let $\varphi \in \mathcal{B}^w_m$ with $w \in \mathcal{M}^\star$. If $\psi = \mathcal{F}_m(\varphi)$ and*

$$\frac{1}{9} \leqslant \left|\frac{\psi^-_{2n}}{\psi^+_{2n}}\right| \leqslant 9,$$

for $|n| \geqslant N_{m,w}$, then

$$|\psi^-_{2n}\psi^+_{2n}| \leqslant |\gamma_n|^2 \leqslant 9|\psi^-_{2n}\psi^+_{2n}|. \quad \rtimes$$

Note that the assumption on ψ is satisfied for φ of real or imaginary type by the symmetry of the adapted Fourier coefficients.

Proof. We begin by writing $\det S_n = g_+ g_-$ with

$$g_\pm \coloneqq \lambda - n\pi - a_n \mp \sigma_n, \qquad \sigma_n = \sqrt{b^+_n b^-_n}.$$

In contrast to the situation of Lemma 1.8, the assumption on ψ effects that $g_\pm$ is continuous and even analytic. Indeed, recall $\psi^\pm_{2n} = b^\pm_n(\alpha_n)$, thus

$$\xi_n \coloneqq \sigma_n(\alpha_n) = \sqrt{b^+_n(\alpha_n) b^-_n(\alpha_n)} \neq 0, \qquad r_n \coloneqq |\xi_n| > 0,$$

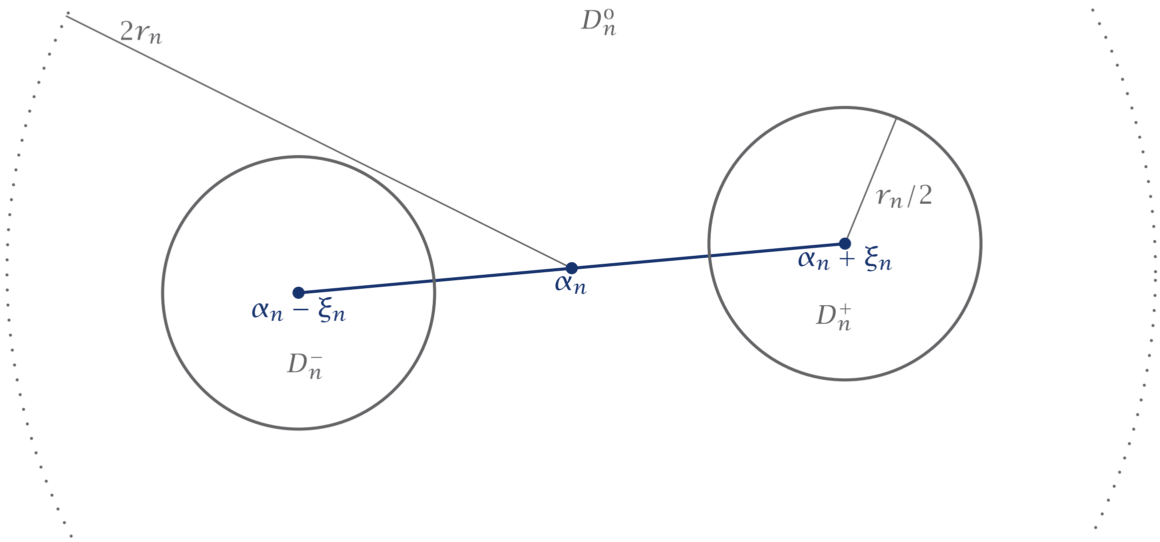

Figure 2.1: The discs $D_n^{\pm}$ contained in D_n°.

so we may choose σ_n as a fixed branch of the square root locally around α_n. Even more, when the disc $D_n^{\circ} \coloneqq \{\lambda : |\lambda - \alpha_n| \leqslant 2r_n\}$ is considered, we have

$$|\partial_\lambda b_n^{\pm}|_{D_n^{\circ}} \leqslant \frac{|b_n^{\pm}|_{U_n}}{\operatorname{dist}(D_n^{\circ}, \partial U_n)} \leqslant \frac{\pi/48}{\pi/2 - |\alpha_n - n\pi| - 2r_n} \leqslant \frac{1}{21},$$

where we used $|\alpha_n - n\pi| \leqslant \pi/48$ and $r_n \leqslant \pi/48$. Note that the same estimate holds for $\partial_\lambda a_n$. Combined with Cauchy's estimate this yields

$$|b_n^{\pm} - b_n^{\pm}(\alpha_n)|_{D_n^{\circ}} \leqslant |\partial_\lambda b_n^{\pm}|_{D_n^{\circ}} 2r_n \leqslant \frac{2}{21} r_n,$$

so $b_n^{\pm}$ does not vanish on D_n° and the function σ_n is analytic there. Hence we can use Rouche's theorem to refine our localization of the roots $\lambda_n^{\pm}$ of $g_{\pm}$ in the following.

By fixing the last two terms of g_+ at α_n, we obtain $h_+ \coloneqq \lambda - n\pi - a_n(\alpha_n) - \sigma_n(\alpha_n)$. We want to compare h_+ and g_+ on the disc

$$D_n^+ \coloneqq \{\lambda : |\lambda - (\alpha_n + \xi_n)| < r_n/2\} \subset D_n^{\circ}.$$

Since $h_+(\alpha_n + \xi_n) = \xi_n - \sigma_n(\alpha_n) = 0$, we have

$$|h_+|_{\partial D_n^+} = |h_+ - h_+(\alpha_n + \xi_n)|_{\partial D_n^+} = \frac{r_n}{2}.$$

In the following lines we will show that on $D_n^{\circ} \supset \overline{D_n^+}$

$$|\partial_\lambda \sigma_n|_{D_n^{\circ}} \leqslant \frac{4}{21}.$$

As a consequence,

$$\begin{aligned}|h_+ - g_+|_{D_n^+} &\leqslant |a_n(\alpha_n) - a_n|_{D_n^\circ} + |\sigma_n(\alpha_n) - \sigma_n|_{D_n^\circ} \\ &\leqslant \left(|\partial_\lambda a_n|_{D_n^\circ} + |\partial_\lambda \sigma_n|_{D_n^\circ}\right) 2r_n < \frac{r_n}{2} = |h_+|_{\partial D_n^+}.\end{aligned}$$

Thus, it follows from Rouche's theorem that g_+ has a unique root contained in D_n^+, while there is no root in $U_n \setminus D_n^+$. In a similar fashion, we find the unique root of g_- to be contained in $D_n^- \coloneqq \{\lambda : |\lambda - (\alpha_n - \xi_n)| < r_n/2\}$. Since the roots of $g_\pm$ are roots of $\det S_n$, they have to coincide with $\lambda_n^\pm$ and hence

$$r_n \leqslant |\lambda_n^+ - \lambda_n^-| \leqslant 3r_n,$$

which is the claim.

It remains to show the estimate for $\partial_\lambda \sigma_n$ on D_n°. To this end, we write

$$|b_n^+| = |b_n^+ b_n^-|^{1/2} \left|\frac{b_n^+}{b_n^-}\right|^{1/2},$$

and recall that

$$\frac{1}{3}r_n \leqslant |b_n^+(\alpha_n)| \leqslant 3r_n, \qquad |b_n^+ - b_n^+(\alpha_n)|_{D_n^\circ} \leqslant \frac{2}{21}r_n.$$

Thus, by the triangle inequality we obtain

$$\frac{5}{21}r_n = \left(\frac{1}{3} - \frac{2}{21}\right) r_n \leqslant |b_n^+|_{D_n^\circ} \leqslant \left(3 + \frac{2}{21}\right) r_n = \frac{65}{21}r_n.$$

Treating b_n^- completely analogous we arrive at

$$\left|\frac{b_n^+}{b_n^-}\right|_{D_n^\circ}, \left|\frac{b_n^-}{b_n^+}\right|_{D_n^\circ} \leqslant 13,$$

which finally yields the desired estimate

$$|\partial_\lambda \sigma_n|_{D_n^\circ} \leqslant \frac{|\partial_\lambda b_n^+|_{D_n^\circ}}{2} \left|\frac{b_n^-}{b_n^+}\right|_{D_n^\circ}^{1/2} + \frac{|\partial_\lambda b_n^-|_{D_n^\circ}}{2} \left|\frac{b_n^+}{b_n^-}\right|_{D_n^\circ}^{1/2} \leqslant \frac{4}{21}. \quad \blacksquare$$

8 Regularity: Potentials of real or imaginary type

We are now in a position to prove Theorem 2. Suppose φ is of real or imaginary type and $\gamma(\varphi) \in h^\nu$ with $\nu \in \mathcal{M}$. The sequence

$$r_n \coloneqq |\varphi_{2n}^+| + |\varphi_{2n}^-| + \nu_{2n}|\gamma_n|, \qquad n \in \mathbb{Z},$$

is clearly square summable, hence by Lemma C.1 there exists an unbounded and monotone weight $w \in \mathcal{M}^\star$ such that $(r_n) \in h^w$. Consequently, $\gamma(\varphi) \in h^{wv}$ and $\varphi \in \mathcal{B}_m^w$ for some $m \geqslant 1$, thus

$$\psi = \mathcal{F}_m(\varphi)$$

is well defined. Since φ is of real or imaginary type, we have by Lemma 1.6

$$b_n^+(\overline{\lambda}) = \pm\overline{b_n^-(\lambda)}.$$

Moreover, $\lambda = \alpha_n(\varphi)$ is real by Lemma 2.2, and therefore

$$|\psi_{2n}^-| = |b_n^-(\alpha_n)| = |b_n^+(\alpha_n)| = |\psi_{2n}^+|.$$

If $\psi_{2n}^+, \psi_{2n}^- \neq 0$, then Lemma 2.6 applies giving

$$|\psi_{2n}^+|^2 = |\psi_{2n}^-|^2 \leqslant |\gamma_n|^2,$$

which is obviously also true in the case $\psi_{2n}^+ = \psi_{2n}^- = 0$. It follows that $\psi \in \mathcal{H}^{wv}$ and Proposition 2.5 yields the claimed regularity of φ.

9 Regularity: Potentials of general type

In the preceding section, we solved the inverse problem for potentials of real or imaginary type. The particular symmetry of the adapted Fourier coefficients of $\psi = \mathcal{F}_m(\varphi)$ allows us to compare them with the gap lengths $\gamma(\varphi)$. By Gasymov's observation, this fails for general potentials. To this end, we consider additional spectral data given by a family of auxiliary gap lengths $\delta(\varphi) = (\delta_n)_{n\in\mathbb{Z}}$, which is characterized by the following three properties:

(a) δ_n is continuously differentiable on some neighbourhood $U \subset \mathcal{H}^w$ of φ,

(b) δ_n vanishes whenever $\lambda_n^+ = \lambda_n^-$ also has a geometric multiplicity of two, and

(c) there exist real numbers ξ_n^+, ξ_n^- and an integer $N_U \geqslant 1$ such that

$$\|\partial\delta_n - t_n\|_0 \leqslant 1/16, \qquad t_n = \frac{1}{2}\left(\mathrm{e}^{2\pi i n(\xi_n^- + \cdot)}, \mathrm{e}^{-2\pi i n(\xi_n^+ + \cdot)}\right),$$

uniformly on U for $|n| \geqslant N_U$.

The properties (a) - (c) are easily verified for $\delta_n = \kappa_n - \tau_n$, with κ_n the nth Dirichlet or Neumann eigenvalue and τ_n the mid-point of the nth spectral gap – see appendix H. These properties enable us to bound the adapted Fourier coefficients of ψ, so we can proceed as in the case of potentials of real or imaginary type, and obtain the general case with no additional costs.

Lemma 2.7 *If $\varphi \in \mathcal{B}_m^w$ with $w \in \mathcal{M}^\star$ has auxiliary gap lengths $\delta(\varphi)$ on some neighbourhood $U \subset \mathcal{B}_m^w$, then there exists an $N \geqslant \max(N_{m,w}, N_U)$, such that*

$$|\delta_n(\varphi) - (\varsigma_n^+ \psi_{2n}^+ + \varsigma_n^- \psi_{2n}^-)| \leqslant \frac{1}{4}(|\psi_{2n}^+| + |\psi_{2n}^-|), \qquad |n| \geqslant N,$$

with $\varsigma_n^+ = \mathrm{e}^{2\pi i n \xi_n^+}/2$ and $\varsigma_n^- = \mathrm{e}^{-2\pi i n \xi_n^-}/2$. ⋊

Proof. We may assume that U is star convex with respect to φ. Put $\psi = \mathcal{F}_m(\varphi)$ and let $V = \mathcal{F}_m(U)$. Then we can choose $N \geqslant \max(N_U, N_{m,w})$ such that for each $|n| \geqslant N$

$$\psi^o := \sum_{k \neq n} (\psi_{2k}^- e_{2k}^- + \psi_{2k}^+ e_{2k}^+) \in V.$$

Fix such an n, then $\varphi^o = \mathcal{F}_m^{-1}(\psi^o) \in U$. Per definitionem, the $2n$th Fourier coefficients of ψ^o vanish, hence $S_n(\varphi^o) \equiv 0$. Consequently, $\alpha_n(\varphi^o)$ is a double periodic eigenvalue of φ^o with a geometric multiplicity of two, so by assumption (b)

$$\delta_n(\varphi^o) = 0.$$

Let $\varphi^s := s\varphi + (1-s)\varphi^o$, then $\varphi^s \in U$ for $0 \leqslant s \leqslant 1$ by star convexity, and hence

$$\begin{aligned} \delta_n(\varphi) &= \delta_n(\varphi) - \delta_n(\varphi^o) \\ &= \int_0^1 \langle \partial \delta_n(\varphi^s), \varphi - \varphi^o \rangle_r \, \mathrm{d}s \\ &= \langle t_n, \varphi - \varphi^o \rangle_r + \langle \theta_n, \varphi - \varphi^o \rangle_r \,, \end{aligned}$$

with $\theta_n = \int_0^1 (\partial \delta_n - t_n)(\varphi^s) \mathrm{d}s$. In addition,

$$\begin{aligned} \varphi - \varphi^o &= \mathcal{F}_m^{-1}(\psi) - \mathcal{F}_m^{-1}(\psi^o) \\ &= \int_0^1 \mathrm{d}\mathcal{F}_m^{-1}(\psi^s)(\psi - \psi^o) \mathrm{d}s \\ &= \psi - \psi^o + \Theta_m(\psi - \psi^o), \end{aligned}$$

with $\Theta_m = \int_0^1 (\mathrm{d}\mathcal{F}_m^{-1} - \mathrm{Id})(\psi^s)\,\mathrm{d}s$. A combination of both representations yields

$$\delta_n(\varphi) = \langle t_n, \psi - \psi^{\mathrm{o}}\rangle_r + \langle t_n, \Theta_m(\psi - \psi^{\mathrm{o}}\rangle_r + \langle \theta_n, \varphi - \varphi^{\mathrm{o}}\rangle_r .$$

Clearly, $\langle t_n, \psi - \psi^{\mathrm{o}}\rangle_r = \psi_{2n}^- \langle t_n^-, \mathrm{e}_{2n}\rangle + \psi_{2n}^+ \langle t_n^+, \mathrm{e}_{-2n}\rangle = \varsigma_n^- \psi_{2n}^- + \varsigma_n^+ \psi_{2n}^+$ and $\|t_n\|_{\mathrm{o}} \leqslant 1$. Moreover, $\|\theta_n\|_{\mathrm{o}} \leqslant 1/16$ by assumption (c), and

$$\|\Theta_m\|_{\mathrm{o}} \leqslant \|\mathrm{d}\mathcal{F}_m^{-1}\|_{\mathrm{o}} \|\mathrm{d}\mathcal{F}_m - \mathrm{Id}\|_{\mathrm{o}} \leqslant \frac{1}{6},$$

by Proposition 2.3. Altogether we obtain

$$\begin{aligned} |\delta_n(\varphi) - \langle t_n, \psi - \psi^{\mathrm{o}}\rangle| &\leqslant \Big(\|t_n\|_{\mathrm{o}}\|\Theta_m\|_{\mathrm{o}} + \|\theta_n\|_{\mathrm{o}}(1 + \|\Theta_m\|_{\mathrm{o}})\Big)\|\psi - \psi^{\mathrm{o}}\| \\ &\leqslant \frac{1}{4}\|\psi - \psi^{\mathrm{o}}\|. \quad \blacksquare \end{aligned}$$

With this preparation we can prove Theorem 4. Suppose $\varphi \in \mathcal{H}^w$ with $w \in \mathcal{M}$. Then for each $|n| \geqslant N$ we have according to the preceding Lemma,

$$|\delta_n(\varphi)|^2 \leqslant \frac{5}{4}\Big(|\psi_{2n}^+| + |\psi_{2n}^-|\Big)^2 \leqslant \frac{10}{4}\left(|b_n^+|_{U_n}^2 + |b_n^-|_{U_n}^2\right).$$

By the same line of arguments as in the proof of Theorem 1, we get $\delta(\varphi) \in h^w$.

Conversely suppose $\varphi \in \mathcal{L}^2$ and $\gamma(\varphi) \in h^v$ as well as $\delta(\varphi) \in h^v$ with $v \in \mathcal{M}$. So

$$r_n \coloneqq |\varphi_{2n}^+| + |\varphi_{2n}^-| + v_{2n}|\gamma_n(\varphi)| + v_{2n}|\delta_n(\varphi)|$$

is an ℓ^2-sequence, hence $(r_n) \in h^w$ for some weight $w \in \mathcal{M}^\star$. Consequently, $\gamma(\varphi) \in h^{vw}$, $\delta(\varphi) \in h^{vw}$ and $\varphi \in \mathcal{B}_m^w$ for some $m \geqslant 1$, thus $\psi = \mathcal{F}_m(\varphi)$ is well defined.

For any $|n| \geqslant N$, with the hypothesis of Lemma 2.6 satisfied, we have

$$|\psi_{2n}^+|^2, |\psi_{2n}^-|^2 \leqslant |\gamma_n(\varphi)|^2.$$

Otherwise, we may assume that $|\psi_{2n}^+| \geqslant 9|\psi_{2n}^-|$, such that by the preceding Lemma

$$\begin{aligned} |\delta_n(\varphi)| &\geqslant |\varsigma_n^+\psi_{2n}^+ + \varsigma_n^-\psi_{2n}^-| - \frac{1}{4}\Big(|\psi_{2n}^+| + |\psi_{2n}^-|\Big) \\ &\geqslant \frac{1}{2}\frac{8}{9}|\psi_{2n}^+| - \frac{1}{4}\frac{10}{9}|\psi_{2n}^+| \\ &\geqslant \frac{1}{6}|\psi_{2n}^+|. \end{aligned}$$

We conclude that in this case

$$\frac{1}{36}|\psi_{2n}^+|^2, |\psi_{2n}^-|^2 \leqslant |\delta_n(\varphi)|^2,$$

so $\psi \in \mathcal{H}^{vw}$. The claim now follows with Proposition 2.5.

Chapter 3

Extensions

10 A note on non-monotone weights

For the sake of brevity and clarity we restricted ourselves to monotone weights. However, this has been a purely technical restriction and Theorem 1-5 remains valid when we drop this requirement and consider weights that are just submultiplicative. This is due to the fact that for a given potential $\varphi \in \mathcal{H}^{\tilde{w}}$, with $\tilde{w} \in \mathcal{M}$ an arbitrary weight, we can always find an unbounded and monotone weight $w \in \mathcal{M}^\star$, such that $\|\varphi\|_{w\tilde{w}} < \infty$ – see Lemma C.1. This enables us to reuse our initial approach by considering weights $v = w\tilde{w} \in \mathcal{M}$ given as the product of an unbounded monotone weight w and an arbitrary other weight $\tilde{w}$. Since we do not use the »other« weight $\tilde{w}$ explicitly, we introduce the following notation to suppress the product structure:

$$v \succ w \quad :\Leftrightarrow \quad v = w\tilde{w} \quad \text{for some} \quad \tilde{w} \in \mathcal{M}.$$

Note that $v \succ w$ implies $v \geqslant w$. Now, given a potential $\varphi \in \mathcal{H}^v$ we may always assume that $v \succ w$ with an unbounded and monotone weight w.

In our treatment of the forward problem in chapter 1 we first used the monotonicity of the weight to obtain an estimate for $\|T_n^2\|_{w;n}$ (Lemma 1.3) and second to show that $b_n^\pm$ is h^w-close to $\varphi_{2n}^\pm$ (Lemma 1.9). We can easily adapt the proofs to the situation of a weight $v \succ w$ with w monotone and unbounded.

Lemma 3.1 *Suppose $w \in \mathcal{M}^\star$. Then for any weight $v \succ w$ and any $n \in \mathbb{Z}$,*

$$\|T_n^2\|_{v;n} \leqslant 8\|\varphi\|_v \left(\frac{\|\varphi\|_v}{\sqrt{|n|}} + \frac{\|R_n\varphi\|_v}{w_n} \right) \leqslant \frac{16}{\hat{w}_n}\|\varphi\|_v^2. \quad \rtimes$$

Proof. Using the addendum to Lemma I.3 instead of Lemma I.3 itself, the proof is identical to that of Lemma 1.3. ∎

When we fix $m \geqslant 1$ and $w \in \mathcal{M}^\star$, we can choose an $N_{m,w} \geqslant 1$ such that $(\mathrm{Id} - T_n)$ is boundedly invertible on $\mathcal{H}^\nu_*$ for any $|n| \geqslant N_{m,w}$, any $\varphi \in \mathcal{B}^\nu_m$ and any $\nu > w$. It is crucial to note that the threshold $N_{m,w}$ only depends on w and works for every weight $\nu > w$ not necessarily monotone. The subsequent arguments of chapter 1 can be repeated word by word, just the final Lemma needs a slight modification.

Lemma 3.2 *Suppose $\varphi \in \mathcal{B}^\nu_m$ with $\nu \in \mathcal{M}$. If $\nu > w$ with $w \in \mathcal{M}^\star$, then for any $N \geqslant N_{m,w}$*

$$\sum_{|n| \geqslant N} \nu_{2n}^2 |b_n^\pm - \varphi_{2n}^\pm|^2 \leqslant \frac{2^{10}}{\hat{w}_N} \|\varphi\|_\nu^6. \quad \rtimes$$

Proof. Using the addendum to Lemma I.3, we can copy the proof of Lemma 1.9. ∎

To adapt our treatment of the inverse problem in chapter 2 it suffices to fix a weight $w \in \mathcal{M}^\star$ and to replace our general assumption $\nu \geqslant w$ with the assumption $\nu > w$. Then $\nu > w$ can be chosen arbitrarily and does not need to be monotone, since we solely used the monotonicity of w.

These somewhat roundabout arguments are indeed necessary, as a given weight $w \in \mathcal{M}$ may not be equivalent to a monotone weight, that is

$$H^w \neq H^\nu$$

for any monotone weight ν. We want to give an example of such a weight in the following. For $w \in \mathcal{M}$ consider the smallest monotone upper bound

$$\check{w}_n = \max_{0 \leqslant k \leqslant n} w_k,$$

and the greatest monotone lower bound

$$\hat{w}_n = \inf_{k \geqslant n} w_k.$$

Both are submultiplicative weights, when w is submultiplicative, and clearly $\hat{w} \leqslant w \leqslant \check{w}$. If $(\check{w}_n / \hat{w}_n)$ is bounded, then the induced norms of those three weights are equivalent, and thus the generated Sobolev spaces are the same

$$H^{\hat{w}} \equiv H^w \equiv H^{\check{w}}.$$

On the other hand, if $(\check{w}_n / \hat{w}_n)$ is unbounded, we have by Proposition E.1

$$H^{\check{w}} \subsetneq H^{\hat{w}},$$

and consequently, $H^w \neq H^v$ for any monotone weight v.

Such weights with $(\check{w}_n/\hat{w}_n)$ being unbounded are called *strongly oscillating*, and one may ask whether they exist at all. Thus we construct an example in the sequel. Since addition is easier visualized than multiplication, we consider the equivalent case of subadditive weights

$$\tilde{w} = \log w, \qquad w \in \mathcal{M}.$$

Clearly, $\tilde{w} + \tilde{v}$ is subadditive, when $\tilde{w}$ and $\tilde{v}$ are. Moreover, the 2-periodic function

$$\Delta(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1, \\ 2 - x, & 1 < x \leqslant 2, \end{cases}$$

gives rise to the subadditive triangular weights

$$\delta_n^k = k \cdot \Delta(n/2^{k^2-1}), \qquad k \geqslant 1,$$

whose height grows linear with k, while its width is 2^{k^2}.

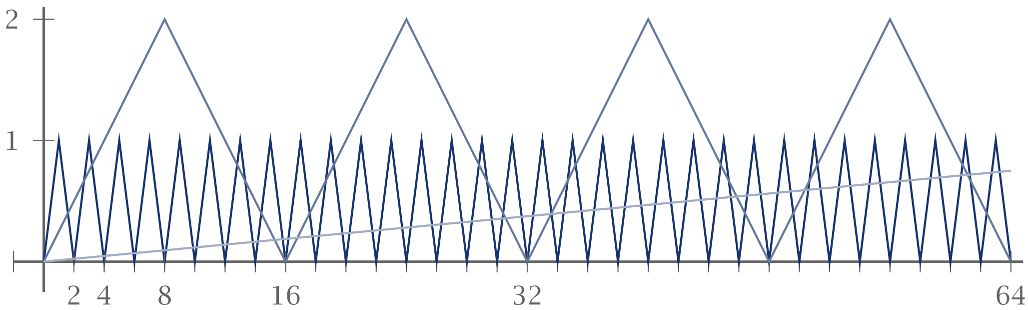

Figure 3.1: The iterations of δ_n^k for $k = 1, \ldots, 3$.

It follows that

$$\tilde{w}_{\mathrm{so}}(n) := \sum_{k \geqslant 1} \delta_n^k$$

is subadditive and clearly unbounded. More to the point,

$$\tilde{w}_{\mathrm{so}}(2^{n^2-1}) = \sum_{k \geqslant n} k \cdot \Delta(2^{n^2-1}/2^{k^2-1}) \geqslant n,$$

while

$$\tilde{w}_{\mathrm{so}}(2^{n^2}) \quad = \sum_{k > n} k \cdot \Delta(2^{n^2}/2^{k^2-1}) = \sum_{l \geqslant 1} \frac{n+l}{2^{l^2+2nl-1}} \leqslant \sum_{l \geqslant 1} \frac{n+l}{4^{nl}} \leqslant \frac{n+1}{4^{n-1}-1} \to 0.$$

So $w_{\mathrm{so}} = \exp(\tilde{w}_{\mathrm{so}})$ is strongly oscillating.

11 Second order gap estimates

In this section, we obtain the first two terms in the asymptotics of the gap lengths $\gamma_n(\varphi)$. They will be revealed by further analyzing the gap length representation known from Lemma 1.8

$$\gamma_n = a_n(\lambda_n^+) - a_n(\lambda_n^-) + \kappa_n^+\sigma_n(\lambda_n^+) + \kappa_n^-\sigma_n(\lambda_n^-), \qquad \sigma_n := \sqrt{b_n^+ b_n^-}, \tag{$\star$}$$

where any configuration of the signs $\kappa_n^+, \kappa_n^- \in \{+1, -1\}$ has to be considered due to possible discontinuities of σ_n.

It turns out that the coefficients $a_n(\lambda_n^\pm)$ do not account to the low order asymptotics of $\gamma_n(\varphi)$, thus we focus on the terms involving σ_n. In particular, we consider the first two terms in the Neumann series expansion of

$$\begin{aligned} b_n^\pm &= \langle (\mathrm{Id} - T_n^\mp T_n^\pm)^{-1}(\varphi_\pm \mathrm{e}_{\mp n}), \mathrm{e}_{\pm n}\rangle \\ &= \varphi_{2n}^\pm + \tilde\varphi_{2n}^\pm + \langle (T_n^\mp T_n^\pm)^2(\mathrm{Id} - T_n^\mp T_n^\pm)^{-1}(\varphi_\pm \mathrm{e}_{\mp n}), \mathrm{e}_{\pm n}\rangle, \end{aligned}$$

where the second term $\tilde\varphi_{2n}^\pm$ depends on λ and is given by

$$\tilde\varphi_{2n}^\pm := \langle T_n^\mp T_n^\pm(\varphi_\pm \mathrm{e}_{\mp n}), \mathrm{e}_{\pm n}\rangle.$$

The first term in the asymptotics of $\gamma_n(\varphi)$ is just $\varphi_{2n}^+\varphi_{2n}^-$, the product of the $2n$th Fourier coefficients of the potential φ, while the second term involves $\varphi_{2n}^\pm$ and $\tilde\varphi_{2n}^\pm$.

Furthermore, it turns out that for $\varphi \in \mathcal{H}^w$ any improvement of the estimates of $\gamma_n(\varphi)$ is limited by the regularity of φ and thus related to the weight w. To express the asymptotics in terms of the growth of w, it is handy to introduce the subclass

$$\mathcal{M}_s := \Big\{ w \in \mathcal{M} \,:\, w \text{ monotone, unbounded and } \sum_{n\neq 0} w_n^2/n^2 < \infty \Big\}.$$

Having fixed notations, we can state the main result of this section.

Theorem 6 *Let $\varphi \in \mathcal{B}_m^v$ with $v \geqslant w$ and $w \in \mathcal{M}_s$. Then*

$$\sum_{|n|\geqslant N_{m,w}} v_{2n}^2 \hat w_n^2 \min_\pm |\gamma_n \pm 2\rho_n|^2 \leqslant C_{m,w}, \qquad \hat w_n = \min\Big(\sqrt{|n|}, w_n\Big),$$

with a real constant $C_{m,w}$ depending only on m and w, and

$$\rho_n := \sqrt{(\varphi_{2n}^+ + \tilde\varphi_{2n}^+)(\varphi_{2n}^- + \tilde\varphi_{2n}^-)}\,\Big|_{\lambda = n\pi}. \quad ⋊$$

When φ is a potential of real type, then the periodic eigenvalues $\lambda_n^\pm$ and the gap lengths γ_n are real. Together with the symmetries of $b_n^\pm$ observed in Lemma 1.6, we obtain an even better estimate in this case.

Supplement to Theorem 6 *Suppose φ is also of real type, then*

$$\sum_{|n|\geqslant N_{m,w}} \nu_{2n}^2 \hat{w}_n^4 |\gamma_n - 2\rho_n|^2 \leqslant C_{m,w}. \quad \rtimes$$

Remark. When the weight is assumed to be of the form $w_n = \langle n\rangle^\delta$ for some $\delta < 1/2$, then the estimate of Theorem 6 can be improved. More to the point, using Hölder's inequality together with the fact that

$$\sum_{n\geqslant 1} \frac{1}{\langle n\rangle^{q\delta}} < \infty, \qquad \sum_{n\geqslant 1} \frac{1}{\langle n\rangle^{p(1-\delta)}} < \infty,$$

for some $p,q \geqslant 1$, one can show that the estimate of $|\gamma_n \pm 2\rho_n|^2$ even holds for $\hat{w}_n^3$ instead of $\hat{w}_n^2$ – see [9]. However, the class $\mathcal{M}_s$ is more general, since it encompasses, among others, weights of the form $\langle n\rangle^{h(n)}$ with $h(n) \uparrow 1/2$ in a slow manner as $n \to \infty$. Therefore, we will not rework the special treatment for weights of the form $\langle n\rangle^\delta$ here. Moreover, when potentials of real type are considered, the approach taken in [9] does does not yield a stronger result than the one presented here. $\multimap$

We begin by expanding γ_n using the representation $(\star)$. This gives

$$\min_\pm |\gamma_n \pm 2\rho_n| \leqslant |a_n(\lambda_n^+) - a_n(\lambda_n^-)| + \min_\pm |\kappa_n^+\sigma_n(\lambda_n^+) + \kappa_n^-\sigma_n(\lambda_n^-) \pm 2\rho_n|.$$

The first summand, called the a_n-term, is easily estimated and does not contribute to the low order asymptotics of γ_n. However, the second summand, called the σ_n-term, requires more attention. We start with the a_n-term.

Lemma 3.3 *For each $\varphi \in \mathcal{B}_m^\nu$ with $\nu \geqslant w$, we have*

$$\sum_{|n|\geqslant N_{m,w}} \nu_{2n}^2 \hat{w}_n^4 |a_n(\lambda_n^+) - a_n(\lambda_n^-)|^2 \leqslant 16m^4 \|\gamma(\varphi)\|_\nu^2. \quad \rtimes$$

Proof. For $|n|$ sufficiently large, the periodic eigenvalues $\lambda_n^\pm$ are located in the disc $D_n \subset U_n$ with $\operatorname{dist}(D_n, \partial U_n) = \pi/3 \geqslant 1$. Hence, Cauchy's estimate F.1 gives

$$|a_n(\lambda_n^+) - a_n(\lambda_n^-)| \leqslant |\partial a_n|_{D_n} |\gamma_n| \leqslant |a_n|_{U_n} |\gamma_n|.$$

The claim now follows with the bound of a_n we found in Lemma 1.7,

$$|a_n|_{U_n} \leqslant \frac{4m^2}{\hat{w}_n^2}, \qquad |n| \geqslant N_{m,w}. \quad \blacksquare$$

We now turn to the σ_n-term. First of all, we improve our localization of the periodic eigenvalues $\lambda_n^\pm$. Given $\varphi \in \mathcal{B}_m^w$, we have by Lemma 1.7

$$|a_n|_{U_n} + |b_n|_{U_n} \leqslant \frac{13m^3}{\hat{w}_n} < \frac{16m^3}{\hat{w}_n}, \qquad |n| \geqslant N_{m,w}.$$

Similar to Lemma 1.8, it follows with Rouche's Theorem that for such n the eigenvalues λ_n^+ and λ_n^- are contained in

$$D_n^* \coloneqq \{\lambda : |\lambda - n\pi| \leqslant r_n\} \subset D_n, \qquad r_n \coloneqq 16m^3/\hat{w}_n.$$

On this even smaller disc, we obtain the following preliminary estimate of the σ_n-term.

Lemma 3.4 *Let $\varphi \in \mathcal{B}_m^\nu$ with $\nu \geqslant w$. Then for each $|n| \geqslant N_{m,w}$*

$$\min_\pm |\kappa_n^+ \sigma_n(\lambda_n^+) + \kappa_n^- \sigma_n(\lambda_n^-) \pm 2\rho_n|^2 \leqslant 36|\sigma_n^2 - \rho_n^2|_{D_n^*}. \quad \rtimes$$

Proof. Put $\xi_n^\pm = \kappa_n^\pm \sigma_n(\lambda_n^\pm)$ and note that we have for each $\lambda \in D_n^*$

$$\min_\pm |\sigma_n(\lambda) \pm \rho_n| \leqslant |\sigma_n^2(\lambda) - \rho_n^2|^{1/2} \leqslant M \coloneqq |\sigma_n^2 - \rho_n^2|_{D_n^*}^{1/2}.$$

Consequently, $\sigma_n(\lambda) = \sqrt{b_n^+(\lambda) b_n^-(\lambda)}$ is for such λ contained in either of the two possibly intersecting discs

$$C_n^+ = \{z : |z - \rho_n| \leqslant M\}, \qquad C_n^- = \{z : |z - (-\rho_n)| \leqslant M\},$$

but may jump between them when λ varies due to possible discontinuities of the root.

Suppose $|\rho_n| \leqslant 2M$. When ξ_n^+ and ξ_n^- are contained in the same disc, there is nothing to do, so we may assume the contrary. For $\xi_n^+ \in C_n^\pm$, we then find

$$|\xi_n^- - (\pm\rho_n)| \leqslant |\xi_n^- - (\mp\rho_n)| + 2|\rho_n| \leqslant 5M.$$

Hence the the claim follows under this assertion on ρ_n.

Conversely suppose $|\rho_n| > 2M$, so the two discs $C_n^\pm$ are disjoint. Moreover,

$$|b_n^+ b_n^- - \rho_n^2|_{D_n^*} \leqslant M^2 < \frac{|\rho_n|^2}{4}$$

by our choice of M. For each $\lambda \in D_n^*$, thus $b_n^+(\lambda) b_n^-(\lambda)$ is contained in the disc of radius $|\rho_n^2|/4$ around ρ_n^2 and never vanishes. So we can choose a fixed branch of the square root to effect that $\sigma_n = \sqrt{b_n^+ b_n^-}$ is analytic on all of D_n^*. It follows from Lemma 1.8 that $\kappa_n^+ = \kappa_n^- = 1$, hence $\xi_n^\pm = \sigma_n(\lambda_n^\pm)$. Let $[\lambda_n^-, \lambda_n^+] \subset D_n^*$ denote the straight line segment connecting λ_n^- and λ_n^+ in the complex plane. By continuity $\sigma([\lambda_n^-, \lambda_n^+])$ is either contained in C_n^+ or in C_n^- and cannot jump. Thus ξ_n^+ and ξ_n^- are contained in the same disc, which gives the claim. ∎

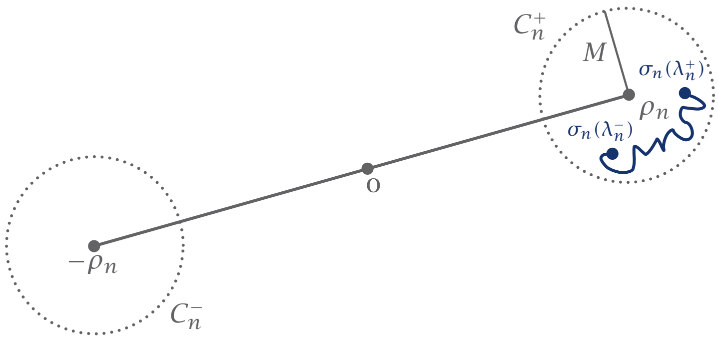

Figure 3.2: $\sigma_n([\lambda_n^-, \lambda_n^+])$ **in the case** $|\rho_n| > 2M$**.**

With this preparation estimating the σ_n-term reduces to estimating $|\sigma_n^2 - \rho_n^2|_{D_n^*}$, which is fairly straightforward. Recall that $\rho_n^2 = (\varphi_{2n}^+ + \tilde{\varphi}_{2n}^+)(\varphi_{2n}^- + \tilde{\varphi}_{2n}^-)\big|_{\lambda=n\pi}$. To obtain a comparable representation of σ_n^2 and ρ_n^2 we define

$$\tilde{b}_n^\pm \coloneqq b_n^\pm - \varphi_{2n}^\pm, \qquad \eta_n^\pm \coloneqq \tilde{\varphi}_{2n}^\pm\big|_{\lambda=n\pi}.$$

Then we may write

$$\sigma_n^2 = (\varphi_{2n}^+ + \tilde{b}_n^+)(\varphi_{2n}^- + \tilde{b}_n^-), \qquad \rho_n^2 = (\varphi_{2n}^+ + \eta_n^+)(\varphi_{2n}^- + \eta_n^-),$$

from what we obtain the following expansion

$$\sigma_n^2 - \rho_n^2 = \varphi_{2n}^+(\tilde{b}_n^- - \eta_n^-) + \varphi_{2n}^-(\tilde{b}_n^+ - \eta_n^+) + \tilde{b}_n^+\tilde{b}_n^- - \eta_n^+\eta_n^-.$$

The estimates needed to attack this expansion are collected in the following lemma.

Lemma 3.5 *For each $\varphi \in \mathcal{B}_m^\nu$ with $\nu \geqslant w$ and $w \in \mathcal{M}_s$ we have*

$$\sum_{|n|\geqslant N_{m,w}} \nu_{2n}^2 w_n^2 |\tilde{b}_n^\pm|_{U_n}^2 \leqslant 2^{10} C_w m^6,$$

with a real constant C_w depending only on the weight w. Moreover,

$$\sum_{|n|\geqslant N_{m,w}} \nu_{2n}^2 \hat{w}_n^4 |\tilde{b}_n^\pm - \tilde{\varphi}_{2n}^\pm|_{U_n}^2 \leqslant 2^{18} C_w m^{10}. \quad \rtimes$$

Proof. We proceed as in Proposition 1.9 and let $u_n = (\mathrm{Id} - T_n^+ T_n^-)^{-1}(\varphi_- \mathrm{e}_n)$. Then (u_n) is a sequence of H^ν-functions which is uniformly bounded by

$$\|u_n\|_{\nu;-n} \leqslant \|(\mathrm{Id} - T_n^+ T_n^-)^{-1}\|_{\nu;-n} \|\varphi_- \mathrm{e}_n\|_{\nu;-n} \leqslant 2m.$$

Moreover, $\tilde{b}_n^- = \langle T_n^+ T_n^- u_n, \mathrm{e}_{-n}\rangle$, so we consider

$$\nu_{2n}^2 w_n^2 |\tilde{b}_n^-|^2 = \nu_{2n}^2 w_n^2 |\langle T_n^+ T_n^- u_n, \mathrm{e}_{-n}\rangle|^2,$$

and follow the argumentation of Lemma I.3 by splitting up the latter term into $\Sigma_{k;n}$, $\Sigma_{l;n}$ and $\Sigma_{\natural;n}$. Since $w \in \mathcal{M}_s$ it is $\sum_{n\neq 0} w_n^2/|n|^2 \leqslant C_w < \infty$. Thus we obtain by exactly the same arguments as in the proof of the lemma

$$\sum_{|n|\geqslant N_{m,w}} \left(\Sigma_{k;n}^2 + \Sigma_{l;n}^2\right) \leqslant 2^7 C_w m^6, \qquad \sum_{|n|\geqslant N_{m,w}} \Sigma_{\natural;n}^2 \leqslant 2^6 m^6.$$

Summing up yields the first claim

$$\sum_{|n|\geqslant N_{m,w}} \nu_{2n}^2 w_n^2 |\tilde{b}_n^\pm|_{U_n}^2 \leqslant 2^{10} C_w m^6.$$

To obtain the second claim, let $u_n^\star = T_n^+ T_n^- (\mathrm{Id} - T_n^+ T_n^-)^{-1}(\varphi_- \mathrm{e}_n)$. Clearly $(u_n^\star)$ is a sequence of H^ν-functions, with individual bound

$$\|u_n^\star\|_{\nu;-n} \leqslant \frac{16}{\hat{w}_n} \|\varphi\|_\nu^2 \|(\mathrm{Id} - T_n^+ T_n^-)^{-1}\|_{\nu;-n} \|(\varphi_- \mathrm{e}_n)\|_{\nu;-n} \leqslant \frac{32 m^3}{\hat{w}_n}$$

for each n. Moreover, $\tilde{b}_n^- - \tilde{\varphi}_{2n}^- = \langle T_n^+ T_n^- u_n^\star, \mathrm{e}_{-n}\rangle$. Repeating the arguments above word by word we arrive at

$$\sum_{|n|\geqslant N_{m,w}} \nu_{2n}^2 \hat{w}_n^4 |\tilde{b}_n^- - \tilde{\varphi}_{2n}^-|_{U_n}^2 \leqslant 2^{18} C_w m^{10}. \quad \blacksquare$$

We are now in a position to proof Theorem 6. First of all, we apply Lemma 3.4 to obtain

$$\frac{1}{2}\nu_{2n}^2 \hat{w}_n^2 \min_\pm |\gamma_n \pm 2\rho_n|^2 \leqslant \nu_{2n}^2 \hat{w}_n^2 |a_n|_{U_n}^2 |\gamma_n|^2 + 36 \nu_{2n}^2 \hat{w}_n^2 |\sigma_n^2 - \rho_n^2|_{D_n^*}.$$

The former term is summable by Lemma 3.3, and for the latter we found the expansion

$$\sigma_n^2 - \rho_n^2 = \varphi_{2n}^+(\tilde{b}_n^- - \eta_n^-) + \varphi_{2n}^-(\tilde{b}_n^+ - \eta_n^+) + \tilde{b}_n^+ \tilde{b}_n^- - \eta_n^+ \eta_n^-.$$

Since $|\eta_n^\pm| \leqslant |\tilde{b}_n^\pm|_{U_n} + |\tilde{b}_n^\pm - \tilde{\varphi}_{2n}^\pm|_{U_n}$, the square terms may be estimated by

$$\begin{aligned}&\frac{1}{2}\nu_{2n}^2 \hat{w}_n^2 (|\tilde{b}_n^+ \tilde{b}_n^-| + |\eta_n^+ \eta_n^-|) \\ &\qquad \leqslant \nu_{2n}^2 \hat{w}_n^2 (|\tilde{b}_n^+|_{U_n}^2 + |\tilde{b}_n^-|_{U_n}^2 + |\tilde{b}_n^+ - \tilde{\varphi}_{2n}^+|_{U_n}^2 + |\tilde{b}_n^- - \tilde{\varphi}_{2n}^-|_{U_n}^2).\end{aligned}$$

Thus they are summable by Lemma 3.5. It remains to consider the mixed terms,

$$\begin{aligned}&\nu_{2n}^2 \hat{w}_n^2 |\varphi_{2n}^+(\tilde{b}_n^- - \eta_n^-) + \varphi_{2n}^-(\tilde{b}_n^+ - \eta_n^+)|_{U_n} \\ &\qquad \leqslant \nu_{2n}^2 (|\varphi_{2n}^+|^2 + |\varphi_{2n}^-|^2) + \nu_{2n}^2 \hat{w}_n^4 (|\tilde{b}_n^+ - \eta_n^+|_{U_n}^2 + |\tilde{b}_n^- - \eta_n^-|_{U_n}^2).\end{aligned}$$

The former term is clearly summable, while an application of Hadamard's lemma gives for the latter

$$\begin{aligned}\tilde{b}_n^{\pm} - \eta_n^{\pm} &= \tilde{b}_n^{\pm} - \tilde{\varphi}_{2n}^{\pm} + \tilde{\varphi}_{2n}^{\pm} - \tilde{\varphi}_{2n}^{\pm}\big|_{\lambda=n\pi} \\ &= \tilde{b}_n^{\pm} - \tilde{\varphi}_{2n}^{\pm} + \int_0^1 \partial\tilde{\varphi}_{2n}^{\pm}((1-s)n\pi + s\lambda)(\lambda - n\pi)\,\mathrm{d}s.\end{aligned}$$

Since $|\lambda - n\pi| \leqslant r_n = 16m^3/\hat{w}_n$ and $|\partial\tilde{\varphi}_{2n}^{\pm}|_{D_n^*} \leqslant |\tilde{\varphi}_{2n}^{\pm}|_{U_n}$ by Cauchy's estimate, we get

$$\frac{1}{32m^3}\nu_{2n}^2\hat{w}_n^4|\tilde{b}_n^{\pm} - \eta_n^{\pm}|^2 \leqslant \nu_{2n}^2\hat{w}_n^4|\tilde{b}_n^{\pm} - \tilde{\varphi}_{2n}^{\pm}|^2 + \nu_{2n}^2\hat{w}_n^2|\tilde{\varphi}_{2n}^{\pm}|^2,$$

so the mixed terms are summable by Lemma 3.5.

Altogether we obtain

$$\sum_{|n|\geqslant N_{m,w}} \nu_{2n}^2\hat{w}_n^2|\sigma_n^2 - \rho_n^2|_{D_n^*} \leqslant C_{w,m},$$

with a real constant $C_{m,w}$ only depending on m and the weight w. This completes the proof of the theorem.

When φ is of real type, the eigenvalues λ_n^+ and λ_n^- are real. Moreover, we infer from the symmetry $b_n^+(\overline{\lambda}) = \overline{b_n^-(\lambda)}$ that $\kappa_n^+ = \kappa_n^- = 1$ as well as

$$\sigma_n(\lambda_n^{\pm}) = \sqrt{b_n^+(\lambda_n^{\pm})b_n^-(\lambda_n^{\pm})} = |b_n^+(\lambda_n^{\pm})|,$$

and similarly

$$\rho_n = \sqrt{(\varphi_{2n}^+ + \eta_n^+)(\varphi_{2n}^- + \eta_n^-)} = |\varphi_{2n}^+ + \eta_n^+|.$$

Thus $|\sigma_n(\lambda_n^+) + \sigma_n(\lambda_n^-) - 2\rho_n| \leqslant 2|\tilde{b}_n^+ - \eta_n^+|_{D_n^*}$, which gives the supplement to Theorem 6.

Chapter 4

Weighted Sobolev spaces

The main concern of this thesis is to compare the regularity of a 1-periodic L^2-function φ to the asymptotic behaviour of some spectral data given as a bi-infinite sequence $\gamma(\varphi) = (\gamma_n)$. In our case, the regularity of φ is sufficiently described by its Fourier coefficients, so we seek a tool to compare their asymptotic behaviour with that of the data $\gamma(\varphi)$. This is where weighted Sobolev spaces come into play, giving a measure for the decay rate of sequences. In the sequel we collect some basic facts about weighted Sobolev spaces and present a mechanism to construct unbounded monotone weights for a given L^2-function. Together with the compact embedding theorems this allows us to transfer local results on L^2 to uniform results on bounded subsets of some weighted Sobolev space.

A Definitions and examples

Let $L^2 \coloneqq L^2(\mathbb{T}, \mathbb{C})$ denote the standard Hilbert space of L^2-integrable, complex-valued functions on the circle $\mathbb{T} \coloneqq \mathbb{R}/\mathbb{Z}$. A function $u \in L^2$ admits a Fourier series expansion

$$u = \sum_{k\in\mathbb{Z}} u_k \mathrm{e}_{2k}, \qquad \mathrm{e}_{2k}(x) \coloneqq \mathrm{e}^{2\pi i k x}.$$

Definition *A normalized, submultiplicative weight is a function $w\colon \mathbb{Z} \to \mathbb{R}$, with*

$$w_{-n} = w_n, \qquad w_n \geqslant 1, \qquad w_{n+m} \leqslant w_n w_m,$$

for all n and m. The space of all such weights is denoted by $\mathcal{M}$. Moreover, the w-norm of a function $u \in L^2$ is defined through

$$\|u\|_w^2 \coloneqq \sum_{k\in\mathbb{Z}} w_{2k}^2 |u_k|^2,$$

and H^w is the Hilbert space of all such functions with finite w-norm. ⋊

The trivial weight $w \equiv 1$ is denoted by o and

$$H^o = L^2 = \bigcup_{w \in \mathcal{M}} H^w,$$

as every weight is assumed to be at least 1.

Definition *A weight $w \in \mathcal{M}$ is called monotone when*

$$w_n \leqslant w_m, \qquad 0 \leqslant n \leqslant m. \quad \rtimes$$

More precisely, such a weight is monotonically increasing. However, monotonically decreasing weights are bounded and thus generate the space $H^o = L^2$. They do not provide any information about asymptotic behaviour, hence we excluded them from the definition of monotone weights.

Moreover, by submultiplicity,

$$w_n \leqslant w_1^n.$$

For this reason a submultiplicative weight cannot grow faster than exponentially. A more precise measure of growth is given by the following

Definition and Lemma A.1 *The characteristic of a weight $w \in \mathcal{M}$,*

$$\chi(w) := \lim_{n \to \infty} \frac{\log(w_n)}{n},$$

exists, is nonnegative and satisfies $\chi(w) = \inf_n \log(w_n)/n$ [19, No. 98]. $\rtimes$

Weights with different characteristics clearly give rise to different weighted Sobolev spaces, while weights with the same characteristic not necessarily give rise to the same Sobolev space. We discuss the question when two given weights generate the same weighted Sobolev space in appendix E, for now we content ourselves with the characteristic as a first order quantity to distinguish weights. Two important subclasses of weights are the following:

Definition *Let $w \in \mathcal{M}$.*

(a) *If $\chi(w) > 0$, then w is called exponential.*

(b) *If w is a monotone weight, and $\log(w_n)/n$ converges eventually monotonically to zero as $n \to \infty$, then w is called subexponential.* $\rtimes$

Note that there exist weights that are neither exponential nor subexponential.

Examples a. Let $\langle n \rangle \coloneqq 1 + |n|$. Then for any $s \geqslant 0$, the polynomial *Sobolev weight*

$$w_n = \langle n \rangle^s$$

gives rise to the usual Sobolev space H^s.

b. For $s \geqslant 0$ and $a > 0$, the exponential *Abel weights*

$$w_n = \langle n \rangle^s \, \mathrm{e}^{a|n|},$$

give rise to the spaces $H^{s,a}$ of L^2-functions, which can be analytically extended to the open strip $\{z : |\mathrm{Im}\, z| < a/2\pi\}$ of the complex plane with traces in H^s on the boundary lines. Note that the characteristic of such a weight is $\chi(w) = a$ and therefore does not provide any information about the polynomial factor $\langle n \rangle^s$ for $s \geqslant 0$.

For example Weierstrass $\wp$-functions are real analytic, 1-periodic functions and thus belong to some $H^{s,a}$ for $a > 0$ sufficiently small. However, they have poles in the complex plane and thus are not entire functions.

c. In between are, among others, the subexponential *Gevrey weights*

$$w_n = \langle n \rangle^s \, \mathrm{e}^{a|n|^\sigma}, \qquad 0 < \sigma < 1,$$

which give rise to the Gevrey-spaces $H^{r,s,\sigma}$, as well as weights of the form

$$w_n = \langle n \rangle^r \exp\left(\frac{a|n|}{1 + (\log \langle n \rangle)^\alpha} \right),$$

that are lighter than Abel weights but heavier than Gevrey weights.

d. The space of 1-periodic entire functions is given by

$$H^\omega = \bigcap_{a>0} H^{0,a}$$

This one as well as other classes of entire functions may be described by weights with $\chi(w) = \infty$, growing faster than exponentially and thus called *superexponential*. Such weights are not submultiplicative, hence our methods do not apply directly. Still one can use results for exponential weights, to infer results for the superexponential case – see [4, 21]. ■

The examples show that certain weights give rise to certain regularity classes of functions. To compare the regularity of functions in H^w with the asymptotics of spectral data, it is convenient to introduce a counterpart of H^w for sequences.

Definition *The w-norm of a complex-valued sequence $(u_k)_{k\in\mathbb{Z}}$ is given by*

$$\|(u_k)\|_w^2 := \sum_{k\in\mathbb{Z}} w_{2k}^2 |u_k|^2,$$

and h^w denotes the space of all such sequences with finite w-norm. ⋊

In particular for any 1-periodic function $u = \sum_{k\in\mathbb{Z}} u_k \mathrm{e}_{2k}$, we have

$$u \in H^w \Leftrightarrow (u_k) \in h^w.$$

B An inequality for products

The product $u \cdot v$ of two H^w functions u and v does in general not belong to H^w. Therefore, we extend our definition of H^w to L^p-based w-norms, which enables us to estimate such products. The *L^p-w-norm* of a function f is given by

$$\|f\|_{p,w} := \left(\sum_{k\in\mathbb{Z}} w_{2k}^p |f_k|^p \right)^{1/p}.$$

Note that for $p \neq 2$ this definition is different from the usual L^p-norm defined using the integral. In particular Hölder's inequality does not hold in this context, as $u \cdot v$ involves the convolution of the Fourier coefficients of u and v. Instead, we use Young's inequality to estimate the w-norm of products.

Weighted Young inequality B.1 *Suppose $u \in L^p$, $v \in L^q$ and*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Then for any $w \in \mathcal{M}$,

$$\|uv\|_{r,w} \leqslant \|u\|_{p,w} \|v\|_{q,w}.$$ ⋊

Proof. The Fourier coefficients of the product of two functions are given as the convolution of the original Fourier coefficients

$$uv = \sum_{k,l\in\mathbb{Z}} u_{k-l} v_l \,\mathrm{e}_{2k}.$$

Therefore, we have by submultiplicity

$$\begin{aligned}\|uv\|_{r,w} &\leqslant \Big(\sum_{k\in\mathbb{Z}}\Big|\sum_{l\in\mathbb{Z}} w_{2k-2l}\,u_{k-l}\cdot w_{2l}\,v_l\Big|^r\Big)^{1/r}\\ &= \|(w_{2k}u_k)*(w_{2l}v_l)\|_r\\ &\leqslant \|(w_{2k}u_k)\|_p\cdot\|(w_{2k}v_k)\|_q\\ &= \|u\|_{p,w}\|v\|_{q,w},\end{aligned}$$

where the ordinary Young inequality for sequences was applied to obtain the last estimate. ▮

Since we use this inequality repeatedly in the case $r = p = 2$ and $q = 1$, we explicitly note that

$$\|u\cdot v\|_w \leqslant \|u\|_w\|v\|_{1,w}.$$

C Construction of unbounded weights

The Fourier coefficients of a given L^2-function clearly decay to zero. However, this decay may happen arbitrary slow, and there is no a priori estimate on the convergence rate. Therefore, the following result may come to some surprise.

Lemma C.1 *For any L^2-function u, there exists an unbounded and monotone weight w, such that $\|u\|_w$ is finite.* ⋊

We denote the subclass of weights in $\mathcal{M}$ that are monotone and unbounded with

$$\mathcal{M}^\star := \{w\in\mathcal{M} : w_n \nearrow \infty \text{ as } |n|\to\infty\}.$$

A consequence of the lemma above is the sufficiency of this subclass in the sense that

$$H^{\mathrm{o}} = \bigcup_{w\in\mathcal{M}^\star} H^w,$$

while obviously $H^w \subsetneq H^{\mathrm{o}}$ for any $w\in\mathcal{M}^\star$. Before we attack the proof, we want to elaborate a criterion for submultiplicity. Since addition is in general easier than multiplication, we consider the equivalent case of subadditivity. More to the point, the submultiplicity of a weight $v\in\mathcal{M}$ implies the subadditivity of $w = \log(v)$,

$$w_{n+m} = \log(v_{n+m}) \leqslant \log(v_n) + \log(v_m) = w_n + w_m,$$

for all n and m. Conversely, $v = \exp(w)$ is submultiplicative, when w is subadditive.

Lemma C.2 *Suppose $w\colon \mathbb{N} \to \mathbb{R}_+$ is a positive function with w_n nondecreasing and w_n/n nonincreasing. Then w is subadditive.* ⋊

Proof. Let $m, n \geqslant 0$, then we have by positivity and monotonicity of w_n/n,

$$\begin{aligned} w_{n+m} &= \frac{n+m}{n+m} w_{n+m} = n\frac{w_{n+m}}{n+m} + m\frac{w_{n+m}}{n+m} \\ &\leqslant n\frac{w_n}{n} + m\frac{w_m}{m} = w_n + w_m. \end{aligned}$$

It remains to consider w_{n-m} for $0 \leqslant m \leqslant n$. However this to the former case by monotonicity,

$$w_{n-m} \leqslant w_{n+m} \leqslant w_n + w_m. \qquad \blacksquare$$

Proof of Lemma C.1 We may assume that (u_k) has infinitely many nonzero terms, otherwise the w-norm of u is finite for any weight w. Let

$$\rho_n \coloneqq r_n^{-1}, \qquad r_n \coloneqq \left(\sum_{|k|\geqslant n} |u_k|^2\right)^{1/4}.$$

Then ρ_n is clearly positive, increasing and unbounded. Moreover,

$$\rho_n^2(|u_n|^2 + |u_{-n}|^2) = 2\frac{|u_n|^2 + |u_{-n}|^2}{2r_n^2} \leqslant 2\frac{r_n^4 - r_{n+1}^4}{r_n^2 + r_{n+1}^2} = 2(r_n^2 - r_{n+1}^2).$$

From this calculation we get

$$\sum_{n=1}^{N} \rho_n^2(|u_n|^2 + |u_{-n}|^2) \leqslant 2r_1^2 \leqslant 2\|u\|_0,$$

for any $N \geqslant 1$. The function $w\colon \mathbb{N} \to \mathbb{R}$ defined by

$$w_1 \coloneqq \log(1+\rho_1) > 0, \qquad w_n \coloneqq \min\left\{\log(1+\rho_n), \frac{n}{n-1}w_{n-1}\right\} \text{ for } n > 1,$$

is positive, unbounded and satisfies the conditions of the previous lemma, as

$$w_n \leqslant w_{n+1}, \qquad \text{and} \qquad \frac{w_{n+1}}{n+1} \leqslant \frac{w_n}{n}.$$

Thus the symmetric extension of w to $\mathbb{Z}$ given by $w_0 \coloneqq 0$ and $w_{-n} \coloneqq w_n$ for $n \geqslant 1$, is subadditive. It follows that $\tilde{w} = \mathrm{e}^w$ is submultiplicative, nondecreasing, unbounded, and $\|u\|_{\tilde{w}} < \infty$. $\blacksquare$

Unbounded weights are handy in many situations. In particular we have the following result.

Rellich's Theorem C.3 *If $w \in \mathcal{M}^\star$, then H^w is compactly embedded into L^2.* ⋊

Proof. Suppose that $u_n \rightharpoonup u$ in H^w. Clearly, the Fourier coefficients of u_n converge pointwise to those of u, since each $\langle \cdot, \mathrm{e}_k \rangle$ is continuous on H^w.

Moreover, by weak convergence, $\|u_n - u\|_w$ is bounded by some $m \geqslant 1$. Let $\varepsilon > 0$, and choose $K \geqslant 1$, such that $m^2/w_{2K}^2 < \varepsilon$. Then

$$\|u_n - u\|_0^2 = \sum_{|k|<K} |u_k^n - u_k|^2 + \sum_{|k|\geqslant K} |u_k^n - u_k|^2 \leqslant \varepsilon + \frac{\|u_n - u\|_w^2}{w_{2K}^2} \leqslant 2\varepsilon,$$

for all n sufficiently large. ∎

When w is unbounded and monotone, Rellich's Theorem allows us to extend results which hold locally uniformly on L^2 to uniform results on bounded subsets of H^w.

D The Cut-Off Lemma

In section 6 we employ an a priori bound on a given function $u \in H^w$. Pöschel's Cut-Off Lemma [21, Lemma 9] allows us to modify the weight w by cutting off a potential large chunk of $\|u\|_w$ arising from finitely many modes while not affecting the asymptotic behaviour when w is subexponential.

Cut-Off Lemma D.1 *If w is either subexponential or exponential, then*

$$w_\varepsilon = \min(w, v_\varepsilon) \in \mathcal{M}, \qquad v_\varepsilon = \mathrm{e}^{\varepsilon|\cdot|},$$

for all sufficiently small positive ε. ⋊

Proof. If w is exponential, then $w_\varepsilon = v_\varepsilon$ for all sufficiently small positive ε, and thus $w_\varepsilon \in \mathcal{M}$.

Now suppose that w is subexponential and let

$$\tilde{w} = \log w, \qquad \tilde{w}_\varepsilon = \log w_\varepsilon.$$

All we have to show is subadditivity of $\tilde{w}_\varepsilon$ which implies submultiplicity of w_ε. Clearly $\tilde{w}_\varepsilon$ is monotone, as $\tilde{w}$ and $\tilde{v}_\varepsilon$ are. Moreover, $\tilde{w}_n/n$ converges eventually to zero, hence for any $\varepsilon > 0$ sufficiently small there exists an integer $N_\varepsilon \geqslant 1$, such that

$$\frac{\tilde{w}_i}{i} \geqslant \varepsilon > \frac{\tilde{w}_n}{n} > \frac{\tilde{w}_m}{m} \quad \text{for} \quad 1 \leqslant i \leqslant N_\varepsilon < n < m.$$

Consequently,

$$(\tilde{w}_\varepsilon)_n = \begin{cases} \tilde{(}\nu_\varepsilon)_n, & n \leqslant N_\varepsilon, \\ \tilde{w}_n, & N_\varepsilon < n, \end{cases}$$

thus $(\tilde{w}_\varepsilon)_n/n$ is nonincreasing. So Lemma C.2 applies giving the claim. ∎

E Equivalent weights and monotonicity

Two weights $w, \nu \in \mathcal{M}$ giving rise to the same weighted Sobolev space $H^w \equiv H^\nu$ are called *equivalent*. It is the purpose of this section to give a criterion for equivalence.

When both quotients (w_n/ν_n) and (ν_n/w_n) are bounded, the weights w and ν clearly give rise to equivalent norms and thus generate the same spaces. It turns out that the converse is also true.

Proposition E.1 *Two weights $w, \nu \in \mathcal{M}$ are equivalent if and only if both (w_n/ν_n) and (ν_n/w_n) are bounded.* ⋊

Before we attack the proof we provide a basic fact about sequences.

Lemma E.2 *Suppose (a_n) is a real, positive, and unbounded sequence, then there exists a real, positive sequence (b_n), such that*

$$\sum_{n\geqslant 1} b_n < \infty, \qquad \text{and} \qquad \sum_{n\geqslant 1} a_n \cdot b_n = \infty. \quad ⋊$$

Proof. Choose a subsequence a_{n_k}, such that $a_{n_k} \geqslant k^2$ for every $k \geqslant 1$. Put $c_k = a_{n_k}$ and $d_k = k^{-2}$. Clearly, $\sum_{n\geqslant 1} d_k < \infty$, while $c_k \cdot d_k \geqslant 1$ and hence is not summable. We now define

$$b_n := \begin{cases} d_k, & n_k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{n\geqslant 1} b_n = \sum_{k\geqslant 1} d_k < \infty$, while on the other hand

$$\sum_{n\geqslant 1} a_n \cdot b_n = \sum_{k\geqslant 1} c_k \cdot d_k = \infty. \quad ∎$$

Proof of Proposition E.1. We may assume that w and v are unbounded, as otherwise the claim follows immediately. Suppose (v_n/w_n) is unbounded, then, by the preceding Lemma, we find a 1-periodic function $h = \sum_{n\in\mathbb{Z}} h_n e_n \in H^v$ with

$$\sum_{n\in\mathbb{Z}} \frac{v_n^2}{w_n^2} v_n^2 |h_n|^2 = \infty.$$

Let $g = \sum_{n\in\mathbb{Z}} g_n e_n$ with $g_n = v_n h_n / w_n$, then $\|g\|_w = \|h\|_v$ and

$$\|g\|_v^2 = \sum_{n\in\mathbb{Z}} \frac{v_n^2}{w_n^2} v_n^2 |h_n|^2 = \infty.$$

Consequently, $H^w \neq H^v$. ∎

Chapter 5

Background Information

F Analytic maps

We present a very short collection of facts about analytic maps in this appendix, mainly to fix notions and notations. A more complete survey including proofs can be found in [15]. For a nice exposition of infinite dimensional holomorphy see also Dineen [3].

In the following let E and F denote complex Banach spaces, and $U \subset E$ an open neighbourhood.

Definition *A map $f: U \to F$ is said to be differentiable at $q \in U$, if there exists a bounded linear map*

$$d_q f: E \to F,$$

called the derivative of f at q, such that

$$\|f(q+h) - f(q) - d_q f(h)\| = o(\|h\|).$$

When f is differentiable at any $q \in U$, it is said to be analytic on U. ⋊

The derivative $d_q f$ is uniquely determined, and in the analytic case defines a map

$$df: U \to L(E,F), \qquad q \mapsto d_q f,$$

which is again analytic on U.

When E is a Hilbert space and f complex valued, there is yet another interpretation of its derivative. We restrict ourselves to the case $E = L^2([0,1],\mathbb{C})$ with $f: U \to \mathbb{C}$ an analytic function on some neighbourhood $U \subset L^2$. For each $q \in U$ the derivative $d_q f$ is a bounded linear functional on L^2. Thus, by the Riesz representation theorem, there exists a uniquely determined function $\partial_q f \in L^2$, such that

$$\mathrm{d}_q f(h) = \langle h, \partial_q f \rangle_r := \int_0^1 h \cdot \partial_q f \,\mathrm{d}x$$

for every $h \in L^2$. We call $\partial_q f$ the *gradient of f at q*, and denote the gradient map of f by

$$\partial f : U \to E, \qquad q \mapsto \partial_q f.$$

The derivative of an analytic function f can be expressed as an integral involving only f itself by Cauchy's formula. An important application is the following estimate, which enables us to bound the norm of the derivative of f in terms of the norm of f, where

$$|f|_U := \sup_{q \in U} |f(q)|_F, \qquad |\mathrm{d}f|_U := \sup_{q \in U} |\mathrm{d}f_q|_{L(E,F)}.$$

Cauchy's estimate F.1 *Let $f : U \to F$ be analytic and $V \subset U$ an open subset. Then*

$$|\mathrm{d}f|_V \leqslant \frac{|f|_U}{\operatorname{dist}(V, \partial U)}. \quad \rtimes$$

Another notable fact is that the uniform limit of analytic maps is again analytic. We only need the following weaker version for power series.

Proposition F.2 *Suppose $f : U \to \mathbb{C}$ is given as a power series of homogeneous polynomials*

$$f = \sum_{k \geqslant 0} P_k.$$

If the power series converges absolutely and uniformly on any compact subset of U, then f is analytic on U. $\rtimes$

Furthermore, when $f : U \to H$ maps from an open subset of a complex Banach space into a Hilbert space, we have the following characterization of analyticity.

Proposition F.3 *Let $f : U \to H$ map into a Hilbert space H with orthonormal basis (e_n). Then f is analytic on U if and only if f is locally bounded, and each coordinate function*

$$f_n = \langle f, e_n \rangle : U \to \mathbb{C}, \qquad n \in \mathbb{Z},$$

is analytic on U. $\rtimes$

Finally we provide a version of the inverse function theorem for near-identity analytic maps. To simplify notation we define

$$\mathcal{B}_m := \{x \in E : |x| \leqslant m\}, \qquad m \geqslant 0,$$

to be the ball of radius m centered at the origin, and let $|\cdot|_m := |\cdot|_{\mathcal{B}_m}$.

Inverse function theorem F.4 *Suppose $f\colon \mathcal{B}_m \to E$ is analytic, and*

$$|f - \mathrm{id}|_m \leqslant m/k, \qquad k > 4.$$

Then f is an analytic diffeomorphism onto its image, and this image covers $\mathcal{B}_{m/2}$. ⋊

Proof. Let F denote the space of analytic functions $g\colon \mathcal{B}_{m/2} \to \mathcal{B}_m$ with $|g - \mathrm{id}|_{m/2} \leqslant m/k$ endowed with the standard metric. Clearly, the operator

$$Tg = \mathrm{id} - (f - \mathrm{id}) \circ g$$

maps F into F, since

$$|Tg - \mathrm{id}|_{m/2} = |(f - \mathrm{id}) \circ g|_{m/2} \leqslant |f - \mathrm{id}|_m \leqslant m/k.$$

Moreover, T is a contraction on F as

$$\begin{aligned} |Tg - Th|_{m/2} &\leqslant \int_0^1 \left|(\mathrm{d}f - \mathrm{Id})((1-s)g + sh)\right|_{m/2} \mathrm{d}s\, |f - g|_{m/2} \\ &\leqslant |\mathrm{d}f - \mathrm{Id}|_{m/k + m/2} |g - h|_{m/2}, \end{aligned}$$

and by Cauchy's estimate

$$|\mathrm{d}f - \mathrm{Id}|_{m/k+m/2} \leqslant \frac{|f - \mathrm{id}|_m}{m/2 - m/k} \leqslant \frac{1/k}{1/2 - 1/k} < 1.$$

The unique fixed point $g\colon \mathcal{B}_{m/2} \to \mathcal{B}_m$ is the analytic inverse to f. ∎

G Spectra

In this appendix we give all necessary definitions and derive some facts about the spectra of Zakharov-Shabat operators

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix},$$

acting on various dense subspaces of $L^2([0,1],\mathbb{C}) \times L^2([0,1],\mathbb{C})$, where $\varphi = (\varphi_-, \varphi_+)$ is a 1-periodic vector potential taken from

$$\mathcal{L}^2 \coloneqq L^2(\mathbb{T},\mathbb{C}) \times L^2(\mathbb{T},\mathbb{C}), \qquad \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

A detailed exposition containing proofs of the facts provided here can be found in [22].

The *periodic spectrum* of $L(\varphi)$ is defined with respect to the dense domain

$$\mathcal{D}_{\mathrm{per}} \coloneqq \left\{ f \in H_c^1 : f(0) = \pm f(1) \right\},$$

where $H_c^1 = H^1([0,1],\mathbb{C}) \times H^1([0,1],\mathbb{C})$ denotes the standard Sobolev space of complex 2-vector valued functions, which together with their derivative are L^2-integrable. The periodic spectrum is pure point and can be obtained as the zero set of an entire function, determined by the trace of the fundamental solution of $L(\varphi)$. The periodic spectrum more precisely [22, Proposition 6.7] consists of a sequence of pairs of complex eigenvalues $\lambda_n^+(\varphi)$ and $\lambda_n^-(\varphi)$, $k \in \mathbb{Z}$, listed with multiplicities, such that

$$\lambda_n^\pm(\varphi) = n\pi + \ell_n^2,$$

locally uniformly in φ. Here, ℓ_n^2 denotes a generic ℓ^2-sequence, i.e.

$$\sum_{k\in\mathbb{Z}} |\lambda_n^\pm - n\pi|^2 < M,$$

where M can be chosen locally uniformly in φ.

If we employ a lexicographical ordering on the complex numbers by

$$a \preccurlyeq b :\Leftrightarrow \begin{cases} \operatorname{Re} a < \operatorname{Re} b \\ \qquad \text{or} \\ \operatorname{Re} a = \operatorname{Re} b \text{ and } \operatorname{Im} a \leqslant \operatorname{Im} b, \end{cases}$$

then the periodic spectrum of $L(\varphi)$ can be represented as a bi-infinite sequence of complex eigenvalues

$$\dots \preccurlyeq \lambda_{n-1}^+ \preccurlyeq \lambda_n^- \preccurlyeq \lambda_n^+ \preccurlyeq \lambda_{n+1}^- \preccurlyeq \dots,$$

counting them with their algebraic multiplicities, where equality or inequality may occur in every place.

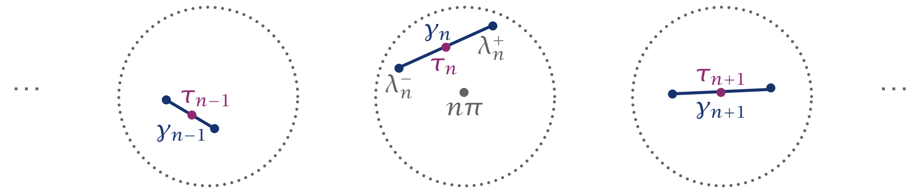

Figure 5.1: The periodic eigenvalues and their spectral gaps for $|n|$ sufficiently large.

The *nth spectral gap length* is defined by

$$\gamma_n \coloneqq \lambda_n^+ - \lambda_n^-,$$

and

$$\tau_n \coloneqq (\lambda_n^+ + \lambda_n^-)/2$$

denotes the *mid-point of the nth gap*. According to [22, Lemma 12.3], τ_n and γ_n^2 are analytic functions for $|n|$ sufficiently large, and their $\mathcal{L}^2$-gradients satisfy

$$\partial\tau_n = \ell_n^2, \qquad \partial\gamma_n^2 = \ell_n^1$$

locally uniformly in φ.

We now turn to the *Dirichlet* and *Neumann spectra*. Those are most transparently defined in AKNS-coordinates [1]. For this reason, we write $\varphi = (q + ip, q - ip)$ to obtain the equivalent representation

$$L(q,p) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} q & p \\ p & -q \end{pmatrix}$$

of $L(\varphi)$, called the *AKNS-system*. The operators $L(\varphi)$ and $L(q,p)$ are unitary equivalent and satisfy

$$L(q,p) = T^{-1}L(\varphi)T, \qquad T = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}.$$

When φ is of real type, q and p can be chosen as real functions, so $L(q,p)$ is real. Hence, $L(\varphi)$ may be viewed as its complexification when q and p are allowed to be complex. In these coordinates the Dirichlet and Neumann spectra of L are defined with respect to the dense domains

$$\mathcal{A}_{\text{Dir}} \coloneqq \left\{ f \in H_c^1 \,:\, f_+(0) = 0 = f_+(1) \right\},$$
$$\mathcal{A}_{\text{Neu}} \coloneqq \left\{ f \in H_c^1 \,:\, f_-(0) = 0 = f_-(1) \right\}.$$

Using the transformation T they are mapped onto the $L(\varphi)$ domains

$$\mathcal{D}_{\text{Dir}} \coloneqq \left\{ g \in H_c^1 \,:\, (g_+ - g_-)\big|_0 = 0 = (g_+ - g_-)\big|_1 \right\},$$
$$\mathcal{D}_{\text{Neu}} \coloneqq \left\{ h \in H_c^1 \,:\, (h_+ + h_-)\big|_0 = 0 = (h_+ + h_-)\big|_1 \right\}.$$

Similar to the periodic case, both spectra are pure point and may be obtained as the zero set of an entire function. The Dirichlet eigenvalues μ_n and the Neumann eigenvalues ν_n are given as bi-infinite unbounded sequences, which can be ordered lexicographically, such that

$$\ldots \preccurlyeq \mu_{n-1} \preccurlyeq \mu_n \preccurlyeq \mu_{n+1} \preccurlyeq \ldots, \qquad \ldots \preccurlyeq \nu_{n-1} \preccurlyeq \nu_n \preccurlyeq \nu_{n+1} \preccurlyeq \ldots.$$

They share the same asymptotics as the periodic eigenvalues, i.e.

$$\mu_n, \nu_n = n\pi + \ell_n^2,$$

locally uniformly in φ – see [22, Lemma 5.3 & 5.5]. When φ is of real type, we even have the following relation

$$\lambda_n^- \leqslant \mu_n, \nu_n \leqslant \lambda_n^+, \qquad n \in \mathbb{Z}.$$

Thus all Dirichlet and Neumann eigenvalues are simple in the real case. Still, μ_n and ν_n are simple for an arbitrary potential when $|n|$ is sufficiently large. Consequently, these eigenvalues are simple roots of their characteristic functions. It follows from the implicit function theorem that they are analytic functions, and by [22, Corollary 7.6] their $\mathcal{L}^2$-gradients satisfy

$$\partial\mu_n = \frac{1}{2}(e_{-2n}^+ + e_{-2n}^-) + \ell_n^2, \qquad \partial\nu_n = -\frac{1}{2}(e_{-2n}^+ + e_{-2n}^-) + \ell_n^2,$$

locally uniformly in φ.

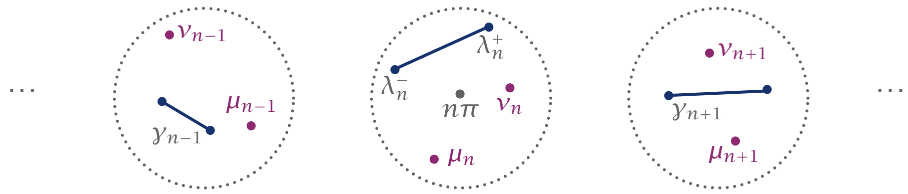

Figure 5.2: Periodic, Dirichlet and Neumann spectra for sufficiently large $|n|$.

H Auxiliary gap lengths

When φ is an arbitrary potential in $\mathcal{H}^w$, the spectral gap lengths $\gamma(\varphi)$ may all vanish and thus may not provide any information about φ. In chapter 2 we thus consider addition spectral data, more precisely, a *family of auxiliary gap lengths $\delta(\varphi) = (\delta_n)_{n\in\mathbb{Z}}$ of φ*, characterized by the following three properties:

(a) δ_n is continuously differentiable on some neighbourhood U of φ,

(b) δ_n vanishes whenever $\lambda_n^+ = \lambda_n^-$ has a geometric multiplicity of two, and

(c) there are real numbers ξ_n^+, ξ_n^- and an integer $N_U \geqslant 1$, such that

$$\|\partial\delta_n - t_n\|_0 \leqslant 1/16, \qquad t_n = \frac{1}{2}\left(\mathrm{e}^{2\pi i n(\xi_n^- + \cdot)}, \mathrm{e}^{-2\pi i n(\xi_n^+ + \cdot)}\right),$$

uniformly on U for $|n| \geqslant N_U$.

Note that the factor $1/2$ in the definition of t_n is due to the usual normalization of the Dirichlet and Neumann eigenfunctions used to express the gradient $\partial\delta_n$ in the case $\delta_n = \mu_n - \tau_n$ and $\delta_n = \nu_n - \tau_n$, respectively. Now, we make up the actual verification of properties (a)-(c) for these two choices of auxiliary gap lengths.

Proposition H.1 *For each $w \in \mathcal{M}^\star$ the quantities $\delta_n = \kappa_n - \tau_n$, with κ_n the nth Dirichlet or Neumann eigenvalue, form a set of auxiliary gap lengths on $\mathcal{B}_m^w$.* ⋊

Proof. Since w is assumed to be monotone and unbounded, the space $\mathcal{H}^w$ is compactly embedded into $\mathcal{L}^2$ by Rellich's Theorem C.3. This allows us to translate results holding locally uniformly on $\mathcal{L}^2$ into uniform results on bounded subsets of $\mathcal{H}^w$. In particular, the balls $\mathcal{B}_m^w$ are compact in $\mathcal{L}^2$. Consequently, the Dirichlet and Neumann eigenvalues satisfy

$$\mu_n, \nu_n = n\pi + \ell_n^2,$$

uniformly on $\mathcal{B}_m^w$. Thus we can choose some $N_\delta \geqslant 1$, such that μ_n and ν_n are simple for all $|n| \geqslant N_\delta$, hence their $\mathcal{L}^2$-gradients exists in every $\varphi \in \mathcal{B}_m^w$, and uniformly satisfy

$$\partial\mu_n = \frac{1}{2}(e_{-2n}^+ + e_{-2n}^-) + \ell_n^2, \qquad \partial\nu_n = -\frac{1}{2}(e_{-2n}^+ + e_{-2n}^-) + \ell_n^2.$$

Eventually increasing N_δ, the same applies to the mid-points τ_n. For $|n|$ sufficiently large they are analytic functions with

$$\partial\tau_n = \ell_n^2,$$

uniformly on $\mathcal{B}_m^w$. Thus $\delta_n = \kappa_n - \tau_n$ is analytic on $\mathcal{B}_m^w$ which gives (a).

Moreover, the gradient of $\delta_n = \mu_n - \tau_n$ satisfies

$$\partial\delta_n = \partial\mu_n - \partial\tau_n = \frac{1}{2}(\mathrm{e}_{2n}, \mathrm{e}_{-2n}) + \ell_n^2,$$

uniformly on $\mathcal{B}_m^w$. So we can choose $\xi_n^+ = \xi_n^- = 0$ and eventually increase N_δ to effect that

$$\|\partial\delta_n - t_n\|_0 \leqslant 1/16, \qquad |n| \geqslant N_\delta,$$

which gives (c) for the Dirichlet case. In a similar fashion one checks property (c) for the Neumann case.

Suppose $\lambda_n^+ = \lambda_n^- \equiv \lambda_n$ with geometric multiplicity two. Then we can choose g as a nontrivial linear combination of functions from the two-dimensional periodic eigenspace of λ_n such that

$$(g_2 - g_1)\big|_0 = 0 = (g_2 - g_1)\big|_1.$$

Thus g is a Dirichlet eigenfunction and $\mu_n = \lambda_n = \tau_n$. The Neumann case is treated similarly, which gives (b) and completes the proof. ∎

I Three auxiliary estimates

Many calculations presented here involve the Dirichlet series

$$\sum_{n\geqslant 1} \frac{1}{n^s}, \qquad s > 1.$$

To make our choice of constants and thresholds convenient, we provide a simple estimate.

Lemma I.1 *Let $N \geqslant 1$ and $s > 1$, then*

$$\sum_{n\geqslant N} \frac{1}{n^s} \leqslant \left(1 + \frac{1}{s-1}\right)\frac{1}{N^{s-1}}, \qquad \sum_{n>N} \frac{1}{n^s} \leqslant \frac{1}{(s-1)N^{s-1}}. \quad ⋊$$

Proof. With $\varphi(x) = (x+N-1)^{-s}$ we may write the sum as

$$\sum_{n\geqslant N} \frac{1}{n^s} = \sum_{n\geqslant 1} \varphi(n).$$

The function φ is positive and monotone decreasing on the interval $[1,\infty)$. Hence,

$$\sum_{n\geqslant N} \frac{1}{n^s} \leqslant \varphi(1) + \int_1^\infty \varphi(x)\,\mathrm{d}x = \frac{1}{N^s} + \frac{1}{(s-1)N^{s-1}}. \quad ∎$$

To be consistent with the definition of w-norms for 2-periodic functions we redefine the w-norm of a given complex valued sequence (u_k) in this appendix as follows

$$\|(u_k)\|_w^2 \coloneqq \sum_{k\in\mathbb{Z}} w_k^2 |u_k|^2.$$

Thus we have

$$u = \sum_{k\in\mathbb{Z}} u_k \mathrm{e}_k \in H_*^w \quad \Leftrightarrow \quad (u_k) \in h^w.$$

Moreover, the i-shifted w-norm of (u_k) is given by

$$\|(u_k)\|_{w;i}^2 = \sum_{k\in\mathbb{Z}} w_{k+i}^2 |u_k|^2.$$

Lemma I.2 *Let $a = (a_k)$ and $x = (x_k)$ be real positive sequences and*

$$h_k \coloneqq \sum_{l\neq n} \frac{a_{k+l}}{|n-k|}\frac{x_{-l}}{|n-l|}, \qquad k \neq n.$$

Then for any monotone weight $w \in \mathcal{M}$ we have

$$\begin{aligned}\|h\|_{1,w;-n} &\leqslant 4\left(\frac{\|a\|_w}{\sqrt{|n|}} + \frac{\|R_n a\|_w}{w_n}\right)\|x\|_{w;-n} \\ &\leqslant \frac{8}{\hat{w}_n}\|a\|_w\|x\|_{w;-n}, \qquad \hat{w}_n \coloneqq \min\left(\sqrt{|n|}, w_n\right). \quad \rtimes\end{aligned}$$

Proof. Our aim is to estimate

$$\|h\|_{1,w;-n} = \sum_{k\neq n} w_{k-n} h_k = \sum_{k,l\neq n} w_{k-n} \frac{a_{k+l}}{|n-k|}\frac{x_{-l}}{|n-l|}.$$

To proceed, we split up the latter sum into two parts $\Sigma_\flat + \Sigma_\natural$ defined by the index sets

$$\begin{aligned} I_\flat &\coloneqq \{k,l\neq n : |n-k| > |n|/2 \quad \text{or} \quad |n-l| > |n|/2\}, \\ I_\natural &\coloneqq \{k,l\neq n : |n-l| \leqslant |n|/2 \quad \text{and} \quad |n-k| \leqslant |n|/2\} = I_\flat^c. \end{aligned}$$

Using Hölder's inequality together with the submultiplicity of the weight w, we obtain for the first term

$$\begin{aligned}\Sigma_\flat &= \sum_{I_\flat} w_{k-n} \frac{a_{k+l}}{|n-k|} \frac{x_{-l}}{|n-l|} \\ &\leqslant \left(\sum_{I_\flat} \frac{1}{|n-k|^2} \frac{1}{|n-l|^2}\right)^{1/2} \left(\sum_{k,l} w_{k+l}^2 a_{k+l}^2 w_{-l-n}^2 x_{-l}^2\right)^{1/2} \\ &\leqslant \frac{4}{\sqrt{|n|}} \|a\|_w \|x\|_{w;-n},\end{aligned}$$

where Lemma I.1 with $s = 2$ was applied in the final step.

Furthermore, for (k,l) taken from $I_\natural$ we have $1 \leqslant |n-k|, |n-l| \leqslant |n|/2$, hence

$$|k+l| \geqslant 2|n| - |n-k| - |n-l| \geqslant |n| \geqslant |n-k|, \qquad |n+l| \geqslant |n|.$$

Thus $w_{k-n} w_n \leqslant w_{k+l} w_{-l-n}$ by the monotonicity of the weight w, and consequently

$$\begin{aligned}\Sigma_\natural &= \sum_{I_\natural} w_{k-n} \frac{a_{k+l}}{|n-k|} \frac{x_{-l}}{|n-l|} \\ &\leqslant \frac{1}{w_n} \left(\sum_{k,l\neq n} \frac{1}{|n-k|^2} \frac{1}{|n-l|^2}\right)^{1/2} \left(\sum_{I_\natural} w_{k+l}^2 a_{k+l}^2 w_{-l-n}^2 x_{-l}^2\right)^{1/2} \\ &\leqslant \frac{4}{w_n} \|R_n a\|_w \|x\|_{w;-n}. \quad \blacksquare\end{aligned}$$

Lemma I.3 *Let $a = (a_k)$, $b = (b_k)$ and $c_n = (c_{k;n})$, for any $n \in \mathbb{Z}$, be real positive sequences, with a uniform bound $\|c_n\|_{w;n} \leqslant M$ for every n. Define*

$$h_{2n} := \sum_{k,l\neq n} \frac{a_{n+l}}{|n-l|} \frac{b_{l+k}}{|n-k|} c_{k;n}.$$

Then for any monotone weight $w \in \mathcal{M}$ and for any $N \geqslant 1$, we have

$$\sum_{|n|>N} w_{2n}^2 h_{2n}^2 \leqslant \frac{2^8}{\hat{w}_N^2} \|a\|_w^2 \|b\|_w^2 M^2, \qquad \hat{w}_N := \min\left(\sqrt{N}, w_N\right). \quad \rtimes$$

Proof. To estimate the w-norm of h, we investigate the weighted coefficients given by

$$w_{2n} h_{2n} = \sum_{k,l\neq n} w_{2n} \frac{a_{n+l}}{|n-l|} \frac{b_{l+k}}{|n-k|} c_{k;n}.$$

For that purpose we define three index sets

$$\begin{aligned} I_k^n &= \{(k,l) : |n-k| > |n|/2\}, \\ I_l^n &= \{(k,l) : |n-l| > |n|/2\}, \\ I_\natural^n &= \{(k,l) : 1 \leqslant |n-k|, |n-l| \leqslant |n|/2\}, \end{aligned}$$

and consider the above expression on each index set separately with the restrictions of the series labeled $\Sigma_{k;n}$, $\Sigma_{l;n}$ and $\Sigma_{\natural;n}$, respectively. In contrast to the situation of Lemma I.2, this time n is not fixed, hence we explicitly note the dependence on n of the Σ terms.

First, Hölder's inequality together with the submultiplicity and symmetry of w gives

$$\begin{aligned}\Sigma_{k;n}^2 &= \left(\sum_{I_k^n} w_{2n} \frac{a_{n+l}}{|n-l|} \frac{b_{l+k}}{|n-k|} c_{k;n}\right)^2 \\ &\leqslant \frac{4}{|n|^2} \left(\sum_{I_k^n} \frac{w_{n+k}^2 c_{k;n}^2}{|n-l|^2}\right)\left(\sum_{k,l} w_{n-k}^2 a_{n+l}^2 b_{l+k}^2\right) \\ &\leqslant \frac{16}{|n|^2} \|c_n\|_{w;n}^2 \|a\|_w^2 \|b\|_w^2.\end{aligned}$$

Second, we find for $\Sigma_{l;n}$ after two applications of Hölder's inequality,

$$\begin{aligned}\Sigma_{l;n}^2 &= \left(\sum_{I_l^n} w_{2n} \frac{a_{n+l}}{|n-l|} \frac{b_{l+k}}{|n-k|} c_{k;n}\right)^2 \\ &\leqslant \frac{4}{|n|^2} \left(\sum_{k} w_{n+k}^2 c_{k;n}^2\right)\left(\sum_{k\neq n} \frac{1}{|n-k|^2} \left(\sum_{l} w_{n-k} a_{n+l} b_{l+k}\right)^2\right) \\ &\leqslant \frac{4}{|n|^2} \|c_n\|_{w;n}^2 \left(\sum_{k\neq n} \frac{1}{|n-k|^2} \left(\sum_{l} w_{n+l}^2 a_{n+l}^2\right)\left(\sum_{l} w_{l+k}^2 b_{l+k}^2\right)\right) \\ &\leqslant \frac{16}{|n|^2} \|c_n\|_{w;n}^2 \|a\|_w^2 \|b\|_w^2.\end{aligned}$$

Together with Lemma I.1 and the uniform bound of $\|c_n\|_{w;n}$ we thus obtain

$$\sum_{|n|>N} \left(\Sigma_{k;n}^2 + \Sigma_{l;n}^2\right) \leqslant 32 \|a\|_w^2 \|b\|_w^2 M^2 \sum_{|n|>N} \frac{1}{|n|^2} \leqslant \frac{64}{N} \|a\|_w^2 \|b\|_w^2 M^2.$$

It remains to consider the term $\Sigma_{\natural;n}$. We start by an application of Hölder, which gives

$$\begin{aligned}\Sigma_{\natural;n}^2 &= \left(\sum_{I_\natural^n} w_{2n} \frac{a_{n+l}}{|n-l|} \frac{b_{l+k}}{|n-k|} c_{k;n}\right)^2 \\ &\leqslant \left(\sum_{k} w_{n+k} c_{k;n}^2\right)\left(\sum_{1\leqslant|n-k|\leqslant|n|/2} \frac{1}{|n-k|^2} \left(\sum_{1\leqslant|n-l|\leqslant|n|/2} \frac{w_{n-k} a_{n+l} b_{l+k}}{|n-l|}\right)^2\right).\end{aligned}$$

Note that for (k,l) with $1 \leqslant |n-k|, |n-l| \leqslant |n|/2$ we have

$$|k+l| \geqslant 2|n| - |n-k| - |n-l| \geqslant |n|, \qquad |n+l| \geqslant |n-k|.$$

Thus by the monotonicity of w, it follows that $w_{n-k}w_n \leqslant w_{n+l}w_{l+k}$, hence

$$\Sigma^2_{\natural;n} \leqslant \frac{1}{w_n^2}\|c\|^2_{w;n}\left(\sum_{k\neq n}\frac{1}{|n-k|^2}\left(\sum_{1\leqslant|n-l|\leqslant|n|/2}\frac{w_{n+l}^2 a_{n+l}^2}{|n-l|^2}\right)\left(\sum_l w_{l+k}^2 b_{l+k}^2\right)\right)$$
$$\leqslant \frac{4}{w_n^2}\|c\|^2_{w;n}\|b\|^2_w\left(\sum_{1\leqslant|m|\leqslant|n|/2}\frac{w_{2n-m}^2 a_{2n-m}^2}{|m|^2}\right),$$

where we changed parameters $n-l \mapsto m$. Since $|2n-m| \geqslant N$ for $|n| \geqslant N$ and $|m| \leqslant |n|/2$, we finally get

$$\sum_{|n|\geqslant N}\sum_{1\leqslant|m|\leqslant|n|/2}\frac{w_{2n-m}^2 a_{2n-m}^2}{|m|^2} \leqslant \sum_{|\nu|\geqslant N}\sum_{|m|\neq 0}\frac{w_\nu^2 a_\nu^2}{|m|^2} \leqslant 4\|a\|^2_w.$$

Consolidating the estimates for $\Sigma_{k;n}$, $\Sigma_{l;n}$ and $\Sigma_{\natural;n}$, we arrive at

$$\sum_{|n|\geqslant N} w_{2n}^2 h_{2n}^2 \leqslant 3\sum_{|n|\geqslant N}\left(\Sigma^2_{k;n}+\Sigma^2_{l;n}+\Sigma^2_{\natural;n}\right)$$
$$\leqslant \frac{2^8}{\hat{w}_N^2}\|a\|^2_w\|b\|^2_w M^2. \qquad \blacksquare$$

For the sake of brevity and clarity we restricted ourselves to monotone weights. Now we consider the general situation. Given a potential $\varphi \in \mathcal{H}^\nu$ with an arbitrary weight $w \in \mathcal{M}$, we can by Lemma C.1 always find an unbounded, monotone weight w such that $\|\varphi\|_{w\cdot\nu} < \infty$. To suppress the product structure in our notation we introduce the notion

$$\nu \gg w \quad :\Leftrightarrow \quad \nu = w\cdot\tilde{w} \ \text{ for some } \ \tilde{w}\in\mathcal{M}.$$

In particular $\nu \gg w$ implies that $\nu \geqslant w$. It turns out that Lemma I.2 and Lemma I.3 can be adapted to fit into this situation.

Addendum to Lemma I.2 *Let $a = (a_k)$ and $x = (x_k)$ be real positive sequences and*

$$h_k := \sum_{l\neq n}\frac{a_{k+l}}{|n-k|}\frac{x_{-l}}{|n-l|}, \qquad k\neq n.$$

Then for any weight $\nu \gg w$ with w monotone, we have

$$\|h\|_{1,\nu;-n} \leqslant 4\left(\frac{\|a\|_\nu}{\sqrt{|n|}}+\frac{\|R_n a\|_\nu}{w_n}\right)\|x\|_{\nu;-n}$$
$$\leqslant \frac{8}{\hat{w}_n}\|a\|_\nu\|x\|_{\nu;-n}, \qquad \hat{w}_n := \min\left(\sqrt{|n|}, w_n\right). \quad \rtimes$$

Proof. We proceed as in the proof of Lemma I.2 and split up the sum into two parts $\Sigma_\flat$ and $\Sigma_\natural$. We did not make use of the monotonicity to estimate $\Sigma_\flat$ hence by exactly the same arguments

$$\Sigma_\flat \leqslant \frac{4}{\sqrt{|n|}} \|a\|_v \|x\|_{v;-n}.$$

By assumption $v \geqslant w$, so vw^{-1} is submultiplicative and symmetric, thus

$$v_{n-k} = w_{n-k} \frac{v_{n-k}}{w_{n-k}} \leqslant w_{n-k} \frac{v_{n+l} v_{l+k}}{w_{n+l} w_{l+k}}.$$

Moreover, for $(k,l) \in I_\natural$ we have $|l+k| \leqslant |n|$ and $|n+l| \geqslant |n-k|$. Hence the monotonicity of w gives

$$v_{n-k} \leqslant \frac{1}{w_{k+l}} \frac{w_{n-k}}{w_{n+l}} v_{n+l} v_{l+k} \leqslant \frac{1}{w_n} v_{n+l} v_{l+k}.$$

Thus we can proceed as in the original proof to obtain

$$\Sigma_\natural \leqslant \frac{4}{w_n} \|R_n a\|_v \|x\|_{v;-n}. \qquad \blacksquare$$

Addendum to Lemma I.3 *Let $a = (a_k)$, $b = (b_k)$ and $c_n = (c_{k;n})$ be, for any $n \in \mathbb{Z}$, real positive sequences, with an uniform bound $\|c_n\|_{v;n} \leqslant M$ for every n. Define*

$$h_{2n} := \sum_{k,l \neq n} \frac{a_{n+l}}{|n-l|} \frac{b_{l+k}}{|n-k|} c_{k;n}.$$

Then for any weight $v \geqslant w$ with w monotone and for any $N \geqslant 1$, we have

$$\sum_{|n|\geqslant N} v_{2n}^2 h_{2n}^2 \leqslant \frac{2^8}{\hat{w}_N^2} \|a\|_v^2 \|b\|_v^2 M^2, \qquad \hat{w}_N := \min\left(\sqrt{N}, w_N\right). \quad \rtimes$$

Proof. We proceed as in the proof of Lemma I.3 by considering the terms $\Sigma_{k;n}$, $\Sigma_{l;n}$ and $\Sigma_{\natural;n}$. To estimate the first two sums we did not make any use of monotonicity, hence we obtain in a similar fashion

$$\sum_{|n|\geqslant N} \left(\Sigma_{k;n}^2 + \Sigma_{l;n}^2\right) \leqslant \frac{64}{N} \|a\|_v^2 \|b\|_v^2 M^2.$$

To estimate the last term $\Sigma_{\natural;n}$, we argument as in the proof of the preceding addendum to obtain for $(k,l) \in I_\natural^n$

$$v_{n-k} \leqslant \frac{1}{w_n} v_{n+l} v_{l+k}.$$

Using this, we can proceed as in the proof of Lemma I.3 to obtain,

$$\begin{aligned}\Sigma_{\natural;n}^2 &\leqslant \|c\|_{\nu;n}\left(\sum_{k\neq n}\frac{1}{|n-k|^2}\left(\sum_{1\leqslant|n-l|\leqslant|n|/2}\frac{\nu_{n-k}^2 a_{n+l} b_{l+k}}{|n-l|}\right)^2\right)\\ &\leqslant \frac{4}{w_n^2}\|c\|_{\nu;n}\|b\|_{\nu}\left(\sum_{1\leqslant|m|\leqslant|n|/2}\frac{\nu_{2n+m}^2 a_{2n+m}^2}{|m|^2}\right).\end{aligned}$$

The conclusion now follows by exactly the same arguments. ∎

The End

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Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne fremde Hilfe bzw. unerlaubte Hilfsmittel angefertigt, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt und die den benutzten Quellen wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

Stuttgart, den 19. Dezember 2011

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